PROBLEM SHEET 7, MP204/274, Semester 1, 1999

(An asterisk $*$ indicates a challenging question.)

1. If A and B are 2×2 matrices, prove that $ch_{AB}(x) = ch_{BA}(x)$.

2. ($*$) If $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times m}(\mathbb{R})$, prove that

$$x^n ch_{AB}(x) = x^m ch_{BA}(x).$$

[Hint: Let C and D be the following $(m+n) \times (m+n)$ partitioned matrices :

$$C = \left[\begin{array}{c|c} xI_m & A \\ \hline B & I_n \end{array}\right], \quad D = \left[\begin{array}{c|c} I_m & 0 \\ \hline -B & xI_n \end{array}\right].$$

Use the equation $\det(CD) = \det(DC)$.]

3. ($*$) Let $A = XY^t$, where $X, Y \in \mathbb{R}^n$ and $X \neq 0, Y \neq 0$. Also let $\lambda = x_1y_1 + \cdots + x_ny_n$.

   (a) Prove that 0 is an eigenvalue of A and that $g_A(0) = n-1$.

   (b) Prove that $AX = \lambda X$.

   (c) If $\lambda \neq 0$, deduce that $ch_A(x) = x^{n-1}(x-\lambda)$ and that A is diagonable.

   (d) If $\lambda = 0$, prove that A is not diagonable.

4. Let $A = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \in M_{3\times 3}(\mathbb{R})$.

   (a) Verify that $ch_A(x) = (x-1)(x+\frac{1}{2})^2$.

   (b) Prove that A is diagonable over $\mathbb{Q}$.

   (c) Find a non–singular matrix $P \in M_{3\times 3}(\mathbb{R})$ such that $P^{-1}AP$ is diagonal.

   (d) Find the decomposition of A into principal idempotents.

   (e) Use (d) to show that $A^n \to \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

5. Let $A \in M_{2\times 2}(\mathbb{R})$ have the property that

$$ch_A = (x-c)(x-\overline{c}),$$

where $c = a+ib, a, b \in \mathbb{R}$ is non–real. Let $X = X_1 + iY_1$, (X_1, Y_1 real vectors) be an eigenvector of A corresponding to the eigenvalue c.

   (i) Prove that $\overline{X} = X_1 - iY_1$ is an eigenvector of A corresponding to the eigenvalue $\overline{c} = a - ib$.

   (ii) If P is the real matrix $P = [X_1|Y_1]$, prove that P is non–singular and that

$$P^{-1}AP = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

   (iii) Find P as defined in (ii), when $A = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$.