1. Let $T : U \to V$ and $S : V \to W$ be linear transformations. Prove that

(a) $\text{rank } ST \leq \text{rank } S$. (Hint: Prove that $\text{Im } ST \subseteq \text{Im } S$.)

(b) $\text{rank } ST \leq \text{rank } T$. (Hint: Prove that $\text{Ker } T \subseteq \text{Ker } ST$.)

(c) If $T$ is surjective then $\text{rank } ST = \text{rank } S$.

(d) If $S$ is injective then $\text{rank } ST = \text{rank } T$.

(e) State corresponding results for matrices.

2. $T : U \to V$ is defined by $T(u_1) = v_1 + 2v_2 + v_3$, $T(u_2) = v_1 + v_2$, where $u_1, u_2$ and $v_1, v_2, v_3$ form bases for $U$ and $V$, respectively. Prove that $T$ is injective but not surjective.

3. $T : V \to V$ is defined by $T(v_1) = v_1 + v_2 + v_3$, $T(v_2) = 2v_1 + v_2 - v_3$, $T(v_3) = v_1 - 2v_3$, where $v_1, v_2, v_3$ form a basis for $V$. Prove that $T$ is not injective and not surjective.

4. $T : U \to V$ is defined by $T(u_1) = v_1 + v_2, T(u_2) = 2v_1 + v_2, T(u_3) = v_1 - v_2$, where $u_1, u_2, u_3$ and $v_1, v_2$ are bases for $U$ and $V$, respectively. Prove that $T$ is surjective but not injective.

5. $T : V \to V$ is defined by $T(v_1) = 2v_1 + v_2, T(v_2) = v_1 - v_2$, where $v_1, v_2$ form a basis for $V$. Prove that $T$ is an isomorphism and calculate $T^{-1}(2v_1 - 3v_2)$.

6. Let $\dim V = 2$ and $T : V \to V$ be a linear transformation such that $T^2 = I_V$.

(a) If $v \in \text{Im } \frac{1}{2}(I_V + T)$, show that $T(v) = v$;

(b) If $v \in \text{Im } \frac{1}{2}(I_V - T)$, show that $T(v) = -v$;

(c) If $T \neq \pm I_V$, show that there are non–zero vectors $v_1$ and $v_2$, such that $T(v_1) = v_1$ and $T(v_2) = -v_2$. Show that these vectors are linearly independent;

(d) If $A$ is a $2 \times 2$ matrix and $A^2 = I_2$ and $A \neq \pm I_2$, show that $A$ is similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(e) Let $A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$. Verify that $A^2 = I_2$ and find a non–singular matrix $P$ such that $P^{-1}AP = \text{diag } (1, -1)$.

7. Let $A = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$. Verify that $P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ satisfies $P^{-1}AP = \text{diag } (3, 1)$ and hence prove that $A^n = \frac{3^n - 1}{2}A + \frac{3 - 3^n}{2}I_2, \quad n \geq 0$.

8. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove that $A^2 - (a + d)A + (ad - bc)I_2 = 0$. 
9. Express the determinant of the matrix

\[ B = \begin{bmatrix}
1 & 1 & 2 & 1 \\
1 & 2 & 3 & 4 \\
2 & 4 & 7 & 2t + 6 \\
2 & 2 & 6 - t & t \\
\end{bmatrix} \]

as a polynomial in \( t \) and hence determine the real values of \( t \) for which \( B^{-1} \) exists.

[Answer: \( \det B = (t - 2)(2t - 1); t \neq 2 \) and \( t \neq \frac{1}{2} \)]

10. Prove that

\[
\begin{vmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
r & r & 1 & 1 \\
r & r & r & 1 \\
\end{vmatrix} = (1 - r)^3.
\]

11. Prove that

\[
\begin{vmatrix}
1 + u_1 & u_1 & u_1 & u_1 \\
u_2 & 1 + u_2 & u_2 & u_2 \\
u_3 & u_3 & 1 + u_3 & u_3 \\
u_4 & u_4 & u_4 & 1 + u_4 \\
\end{vmatrix} = 1 + u_1 + u_2 + u_3 + u_4.
\]