PROBLEM SHEET 6, MP204/274, Semester 1, 1999

1. Let $T : U \to V$ and $S : V \to W$ be linear transformations. Prove that

   (a) $\operatorname{rank} ST \le \operatorname{rank} S$. (Hint: Prove that $\operatorname{Im} ST \subseteq \operatorname{Im} S$.)

   (b) $\operatorname{rank} ST \le \operatorname{rank} T$. (Hint: Prove that $\operatorname{Ker} T \subseteq \operatorname{Ker} ST$.)

   (c) If T is surjective then $\operatorname{rank} ST = \operatorname{rank} S$.

   (d) If S is injective then $\operatorname{rank} ST = \operatorname{rank} T$.

   (e) State corresponding results for matrices.

2. $T : U \to V$ is defined by $T(u_1) = v_1 + 2v_2 + v_3,\ T(u_2) = v_1 + v_2$, where u_1, u_2 and v_1, v_2, v_3 form bases for U and V, respectively. Prove that T is injective but not surjective.

3. $T : V \to V$ is defined by $T(v_1) = v_1 + v_2 + v_3,\ T(v_2) = 2v_1 + v_2 - v_3,\ T(v_3) = v_1 - 2v_3$, where v_1, v_2, v_3 form a basis for V. Prove that T is not injective and not surjective.

4. $T : U \to V$ is defined by $T(u_1) = v_1 + v_2, T(u_2) = 2v_1 + v_2, T(u_3) = v_1 - v_2$, where u_1, u_2, u_3 and v_1, v_2 are bases for U and V, respectively. Prove that T is surjective but not injective.

5. $T : V \to V$ is defined by $T(v_1) = 2v_1 + v_2,\ T(v_2) = v_1 - v_2$, where v_1, v_2 form a basis for V. Prove that T is an isomorphism and calculate $T^{-1}(2v_1 - 3v_2)$.

6. Let $\dim V = 2$ and $T : V \to V$ be a linear transformation such that $T^2 = I_V$.

   (a) If $v \in \operatorname{Im} \frac{1}{2}(I_V + T)$, show that $T(v) = v$;

   (b) If $v \in \operatorname{Im} \frac{1}{2}(I_V - T)$, show that $T(v) = -v$;

   (c) If $T \ne \pm I_V$, show that there are non–zero vectors v_1 and v_2, such that $T(v_1) = v_1$ and $T(v_2) = -v_2$. Show that these vectors are linearly independent;

   (d) If A is a 2×2 matrix and $A^2 = I_2$ and $A \ne \pm I_2$, show that A is similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

   (e) Let $A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$. Verify that $A^2 = I_2$ and find a non–singular matrix P such that $P^{-1}AP = \operatorname{diag}(1, -1)$.

7. Let $A = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$. Verify that $P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ satisfies $P^{-1}AP = \operatorname{diag}(3,\ 1)$ and hence prove that

$$A^n = \frac{3^n - 1}{2} A + \frac{3 - 3^n}{2} I_2, \quad n \ge 0.$$

8. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove that $A^2 - (a + d)A + (ad - bc)I_2 = 0$.

9. Express the determinant of the matrix

$$B = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 2t+6 \\ 2 & 2 & 6-t & t \end{bmatrix}$$

as as polynomial in t and hence determine the real values of t for which B^{-1} exists.

[Answer: $\det B = (t-2)(2t-1)$; $t \neq 2$ and $t \neq \frac{1}{2}$.]

10. Prove that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \\ r & r & 1 & 1 \\ r & r & r & 1 \end{vmatrix} = (1-r)^3.$$

11. Prove that

$$\begin{vmatrix} 1+u_1 & u_1 & u_1 & u_1 \\ u_2 & 1+u_2 & u_2 & u_2 \\ u_3 & u_3 & 1+u_3 & u_3 \\ u_4 & u_4 & u_4 & 1+u_4 \end{vmatrix} = 1 + u_1 + u_2 + u_3 + u_4.$$