PROBLEM SHEET 5, MP204/274, Semester 1, 1999

(A † indicates a challenging question.)
In the following $P_n[\mathbb{R}]$ denotes the vetor space consisting of all polynomials of the form $a_0 + a_1x + \cdots + a_nx^n$, where $a_0, \ldots, a_n \in \mathbb{R}$.

1. Suppose $V = M_{2\times 2}(\mathbb{R})$ and $\beta : E_{11}, E_{12}, E_{21}, E_{22}$ is the standard basis for $V$. Mappings $S, T : V \to V$ are defined by

$$T(A) = \frac{1}{2}(A - A^t),\ S(A) = \frac{1}{2}(A + A^t).$$

   (i) Prove that $S$ and $T$ are linear.

   (ii) Find $[S]_\beta^\beta$ and $[T]_\beta^\beta$.

   (iii) Find bases for $\operatorname{Ker} S$ and $\operatorname{Im} S$, $\operatorname{Ker} T$ and $\operatorname{Im} T$.

   (iv) Prove that $S^2 = S$, $T^2 = T$.

   (v) Prove that $ST = 0$, $TS = 0$.

   (vi) Prove that $S + T = I_V$.

2. Let $\gamma : E_1,\ E_2,\ E_3$ be the usual basis of unit vectors for $V = \mathbb{R}^3$ and let $\beta : v_1,\ v_2,\ v_3$ be the basis of $\mathbb{R}^3$ given by

$$v_1 = [1,\ 1,\ -1]^t,\ v_2 = [2,\ 1,\ 3]^t,\ v_3 = [0,\ 1,\ 1]^t.$$

   Find (i) $[I_V]_\beta^\gamma$ and (ii) $[I_V]_\gamma^\beta$. [Ans: (i) $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$, (ii) $\begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & -1/6 & 1/6 \\ -2/3 & 5/6 & 1/6 \end{bmatrix}$.]

3. Let $A$ and $B$ be non–singular $n \times n$ matrices over $\mathbb{R}$ and let $V = M_{n\times n}(\mathbb{R})$. Show that the mapping $T : V \to V$ defined by $T(X) = AXB$ has the property that $\operatorname{Ker} T = \{0\}$ and $\operatorname{Im} T = V$.

4. A mapping $T : P_2[\mathbb{R}] \to \mathbb{R}^3$ is defined by

$$T(f(x)) = \begin{bmatrix} f(1) \\ f(0) \\ f(-1) \end{bmatrix}.$$

   (a) Prove that $T$ is a linear transformation.

   (b) If $S : \mathbb{R}^3 \to P_2[\mathbb{R}]$ is the linear transformation defined by

$$S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = b + \frac{a-c}{2}x + \frac{a-2b+c}{2}x^2,$$

   verify that $ST = I_{P_2[\mathbb{R}]}$ and $TS = I_{\mathbb{R}^3}$.

5. Let $V = P_3[\mathbb{R}]$ denote the vector space of all polynomials of the form $a + bx + cx^2 + dx^3$. Show that the rule

$$T(f(x)) = \frac{1}{x}\int_0^x f(t)dt - f(x)$$

   defines a linear mapping $T : V \in V$. Find the matrix of $T$ with respect to the standard basis $1, x, x^2, x^3$. Also find bases for $\operatorname{Im} T$ and $\operatorname{Ker} T$.

6. Let $T : P_2[\mathbb{R}] \to P_2[\mathbb{R}]$ be given by $T(f(x)) = f'(x)g(x) + 2f(x)$, where $g(x) = 3 + x$ and $f'(x)$ is the formal derivative of $f$ (i.e. if $f = a_0 + a_1x + a_2x^2$, then $f'(x) = a_1 + 2a_2x$, where $a,\ b,\ c \in \mathbb{R}$).

   Also let $S : P_2[\mathbb{R}] \to \mathbb{R}^3$ be defined by $S(a+bx+cx^2) = [a+b,\ c,\ a-b]^t$, where $a,\ b,\ c \in \mathbb{R}$.

   Let $\beta : 1,\ x,\ x^2$ and $\gamma : e_1,\ e_2,\ e_3$ be the usual bases for $P_2[\mathbb{R}]$ and $\mathbb{R}^3$, respectively.

   Find $[S]_\beta^\gamma$, $[T]_\beta^\beta$ and $[ST]_\beta^\gamma$.

7. Let $T : P_4[\mathbb{R}] \to P_4[\mathbb{R}]$ be the linear transformation defined by

$$T(f(x)) = \frac{1}{2}(f(x) + f(-x)).$$

   (i) Prove that $T^2 = T$.

   (ii) For the basis $\beta : 1,\ x^2,\ x^4,\ x,\ x^3$ of $P_4[\mathbb{R}]$, find $[T]_\beta^\beta$.

8. Let $T : U \to V$ and $S : V \to W$ be linear transformations.

   (a) If Ker $ST = \langle u_1, \ldots, u_n\rangle$, prove that

$$\operatorname{Im} T \cap \operatorname{Ker} S = \langle T(u_1), \ldots, T(u_n)\rangle.$$

   (b) (†) Let $L$ be the restriction of $T$ to $\operatorname{Ker} ST$, that is $L(u) = T(u)$ if $u \in \operatorname{Ker} ST$. (Note: $L : \operatorname{Ker} ST \to V$ is a linear transformation.) Prove that $\operatorname{Im} L = \operatorname{Im} T \cap \operatorname{Ker} S$ and $\operatorname{Ker} L = \operatorname{Ker} T$ and deduce that

$$\dim(\operatorname{Im} T \cap \operatorname{Ker} S) = \operatorname{nullity} ST - \operatorname{nullity} T.$$

   (c) From (b) deduce that

$$\begin{aligned} \operatorname{nullity} ST &\le \operatorname{nullity} S + \operatorname{nullity} T, \\ \operatorname{rank} ST &\ge \operatorname{rank} S + \operatorname{rank} T - \dim V. \end{aligned}$$

   (d) Use (c) to prove that if $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times p}(\mathbb{R})$, then

$$\begin{aligned} \operatorname{nullity} AB &\le \operatorname{nullity} A + \operatorname{nullity} B, \\ \operatorname{rank} AB &\ge \operatorname{rank} A + \operatorname{rank} B - n. \end{aligned}$$

9. If $A$ is $n\times n$ and $A^k = 0$, use a generalisation of part (d) of the previous question to prove $\operatorname{rank} A \le n(1 - \frac{1}{k})$.

10. Let $T : V \to V$ have the property that $T(T(v)) = T(v)$ for all $v \in V$. Prove that

   (a) $\operatorname{Ker} T \cap \operatorname{Im} T = \{0\}$.

   (b) $V = \operatorname{Ker} T + \operatorname{Im} T$.

   (c) Prove that if $u_1, \ldots, u_r$ forms a basis for $\operatorname{Im} T$ and $u_{r+1}, \ldots, u_n$ forms a basis for $\operatorname{Ker} T$, then $u_1, \ldots, u_n$ forms a basis $\beta$ for $V$ and that

$$[T]_\beta^\beta = \operatorname{diag}(\underbrace{1, \ldots, 1}_{r}, \underbrace{0, \ldots, 0}_{n-r}).$$

11. Let $T : U \to V$ and $S : V \to W$ be linear transformations. Prove that

   (a) $\operatorname{rank} ST \leq \operatorname{rank} S$. (Hint: Prove that $\operatorname{Im} ST \subseteq \operatorname{Im} S$.)

   (b) $\operatorname{rank} ST \leq \operatorname{rank} T$. (Hint: Prove that $\operatorname{Ker} T \subseteq \operatorname{Ker} ST$.)

   (c) If $T$ is surjective then $\operatorname{rank} ST = \operatorname{rank} S$.

   (d) If $S$ is injective then $\operatorname{rank} ST = \operatorname{rank} T$.

   (e) State corresponding results for matrices.