In the following $P_n[\mathbb{R}]$ denotes the vector space consisting of all polynomials of the form $a_0 + a_1 x + \cdots + a_n x^n$, where $a_0, \ldots, a_n \in \mathbb{R}$.

1. Suppose $V = M_{2 \times 2}(\mathbb{R})$ and $\beta : E_{11}, E_{12}, E_{21}, E_{22}$ is the standard basis for $V$. Mappings $S, T : V \to V$ are defined by

$$T(A) = \frac{1}{2}(A - A^t), \quad S(A) = \frac{1}{2}(A + A^t).$$

(i) Prove that $S$ and $T$ are linear.
(ii) Find $[S]_\beta^\beta$ and $[T]_\beta^\beta$.
(iii) Find bases for $\text{Ker} S$ and $\text{Im} S$, $\text{Ker} T$ and $\text{Im} T$.
(iv) Prove that $S^2 = S$, $T^2 = T$.
(v) Prove that $ST = 0$, $TS = 0$.
(vi) Prove that $S + T = I_V$.

2. Let $\gamma : E_1, E_2, E_3$ be the usual basis of unit vectors for $V = \mathbb{R}^3$ and let $\beta : v_1, v_2, v_3$ be the basis of $\mathbb{R}^3$ given by

$$v_1 = [1, 1, -1]^t, \quad v_2 = [2, 1, 3]^t, \quad v_3 = [0, 1, 1]^t.$$

Find (i) $[I_V]_\beta^\gamma$ and (ii) $[I_V]_\gamma^\beta$. [Ans: (i) $\left[ \begin{array}{ccc} 1 & 2 & 0 \\
1 & 1 & 1 \\
-1 & 3 & 1 \end{array} \right]$, (ii) $\left[ \begin{array}{ccc} 1/3 & 1/3 & -1/3 \\
1/3 & -1/6 & 1/6 \\
-2/3 & 5/6 & 1/6 \end{array} \right]$]

3. Let $A$ and $B$ be non-singular $n \times n$ matrices over $\mathbb{R}$ and let $V = M_{n \times n}(\mathbb{R})$. Show that the mapping $T : V \to V$ defined by $T(X) = AXB$ has the property that $\text{Ker} T = \{0\}$ and $\text{Im} T = V$.

4. A mapping $T : P_2[\mathbb{R}] \to \mathbb{R}^3$ is defined by

$$T(f(x)) = \left[ \begin{array}{c} f(1) \\
f(0) \\
f(-1) \end{array} \right].$$

(a) Prove that $T$ is a linear transformation.
(b) If $S : \mathbb{R}^3 \to P_2[\mathbb{R}]$ is the linear transformation defined by

$$S \left( \begin{array}{c} a \\
b \\
c \end{array} \right) = b + \frac{a-c}{2} x + \frac{a - 2b + c}{2} x^2,$$

verify that $ST = I_{P_2[\mathbb{R}]}$ and $TS = I_{\mathbb{R}^3}$.

5. Let $V = P_3[\mathbb{R}]$ denote the vector space of all polynomials of the form $a + bx + cx^2 + dx^3$. Show that the rule

$$T(f(x)) = \frac{1}{x} \int_0^x f(t)dt - f(x)$$

defines a linear mapping $T : V \in V$. Find the matrix of $T$ with respect to the standard basis $1, x, x^2, x^3$. Also find bases for $\text{Im} T$ and $\text{Ker} T$. 

6. Let \( T : P_2[\mathbb{R}] \to P_2[\mathbb{R}] \) be given by \( T(f(x)) = f'(x)g(x) + 2f(x) \), where \( g(x) = 3 + x \) and \( f'(x) \) is the formal derivative of \( f \) (i.e. if \( f = a_0 + a_1x + a_2x^2 \), then \( f'(x) = a_1 + 2a_2x \), where \( a, b, c \in \mathbb{R} \).

Also let \( S : P_2[\mathbb{R}] \to \mathbb{R}^3 \) be defined by \( S(a+bx+cx^2) = [a+b, c, a-b]^t \), where \( a, b, c \in \mathbb{R} \).

Let \( \beta : 1, x, x^2 \) and \( \gamma : e_1, e_2, e_3 \) be the usual bases for \( P_2[\mathbb{R}] \) and \( \mathbb{R}^3 \), respectively.

Find \([S]_{\beta}^\gamma, [T]_{\beta}^\gamma\) and \([ST]_{\beta}^\gamma\).

7. Let \( T : P_4[\mathbb{R}] \to P_4[\mathbb{R}] \) be the linear transformation defined by

\[
T(f(x)) = \frac{1}{2}(f(x) + f(-x)).
\]

(i) Prove that \( T^2 = T \).

(ii) For the basis \( \beta : 1, x^2, x^4, x, x^3 \) of \( P_4[\mathbb{R}] \), find \([T]_{\beta}^\gamma\).

8. Let \( T : U \to V \) and \( S : V \to W \) be linear transformations.

(a) If \( \ker ST = \langle u_1, \ldots, u_n \rangle \), prove that

\[
\text{im} \ T \cap \ker S = \langle T(u_1), \ldots, T(u_n) \rangle.
\]

(b) Let \( L \) be the restriction of \( T \) to \( \ker ST \), that is \( L(u) = T(u) \) if \( u \in \ker ST \). (Note: \( L : \ker ST \to V \) is a linear transformation.)

Prove that \( \text{im} \ L = \text{im} \ T \cap \ker S \) and \( \ker L = \ker T \) and deduce that

\[
\text{dim} \ (\text{im} \ T \cap \ker S) = \text{nullity} \ ST - \text{nullity} \ T.
\]

(c) From (b) deduce that

\[
\begin{align*}
\text{nullity} \ ST & \leq \text{nullity} \ S + \text{nullity} \ T, \\
\text{rank} \ ST & \geq \text{rank} \ S + \text{rank} \ T - \text{dim} \ V.
\end{align*}
\]

(d) Use (c) to prove that if \( A \in M_{m \times n}(\mathbb{R}) \) and \( B \in M_{n \times p}(\mathbb{R}) \), then

\[
\begin{align*}
\text{nullity} \ AB & \leq \text{nullity} \ A + \text{nullity} \ B, \\
\text{rank} \ AB & \geq \text{rank} \ A + \text{rank} \ B - n.
\end{align*}
\]

9. If \( A \) is \( n \times n \) and \( A^k = 0 \), use a generalisation of part (d) of the previous question to prove \( \text{rank} \ A \leq n(1 - \frac{1}{k}) \).

10. Let \( T : V \to V \) have the property that \( T(T(v)) = T(v) \) for all \( v \in V \).

Prove that

(a) \( \ker T \cap \text{im} \ T = \{0\} \).

(b) \( \text{im} \ T = \ker T + \text{im} \ T \).

(c) Prove that if \( u_1, \ldots, u_r \) forms a basis for \( \text{im} \ T \) and \( u_{r+1}, \ldots, u_n \) forms a basis for \( \ker T \), then \( u_1, \ldots, u_n \) forms a basis \( \beta \) for \( V \) and that

\[
[T]_{\beta}^\gamma = \text{diag} (1, \ldots, 1, 0, \ldots, 0).
\]
11. Let $T : U \to V$ and $S : V \to W$ be linear transformations. Prove that

(a) $\text{rank } ST \leq \text{rank } S$. (Hint: Prove that $\text{Im } ST \subseteq \text{Im } S$.)
(b) $\text{rank } ST \leq \text{rank } T$. (Hint: Prove that $\text{Ker } T \subseteq \text{Ker } ST$.)
(c) If $T$ is surjective then $\text{rank } ST = \text{rank } S$.
(d) If $S$ is injective then $\text{rank } ST = \text{rank } T$.
(e) State corresponding results for matrices.