PROBLEM SHEET 4, MP204/274, Semester 1, 1999

(An $*$ indicates a challenging question.)

1. A is a 3×3 matrix such that $A^2 = 0$ and $A \neq 0$. Prove that

   (a) $C(A) \subseteq N(A)$. (Hint: Let $X \in C(A)$. Then $X = AY$ for some $Y \in \mathbb{R}^n$.)

   (b) $\operatorname{rank} A = 1$. (Hint: Use the *rank+ nullity* theorem.)

   (c) Exhibit a nonzero 3×3 matrix A such that $A^2 = 0$.

2. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation which maps $(1,2)^t$ to $(-2,3)^t$ and $(1,-1)^t$ to $(5,2)^t$. Find $T(v)$ when $v = (7,5)^t$. (Ans: $7, 18)^t$.)

3. Let $T : U \to V$ be a linear transformation. Using the *rank+ nullity* theorem, prove that

   (a) $\operatorname{rank} T \leq \dim U$.

   (b) $\operatorname{Ker} T = \{0\} \Rightarrow \dim U \leq \dim V$.

4. Let U be a vector space with basis u_1, u_2, u_3. $T : U \to U$ is the linear transformation defined by

$$T(u_1) = u_3, T(u_2) = -u_3, T(u_3) = u_1 + u_2.$$

   Find bases for $\operatorname{Ker} T, \operatorname{Im} T$. Also find $\operatorname{rank} T$ and $\operatorname{nullity} T$.

5. Let U be a vector space with basis u_1, u_2, u_3. $T : U \to U$ is the linear transformation defined by

$$\begin{aligned} T(u_1) &= u_1 + u_2 + u_3 \\ T(u_2) &= u_1 - u_2 + u_3 \\ T(u_3) &= 2u_1 + 2u_3. \end{aligned}$$

   Find bases for $\operatorname{Ker} T, \operatorname{Im} T$. Also find $\operatorname{rank} T$ and $\operatorname{nullity} T$.

6. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and let $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ be the linear transformation defined by $T(X) = AX - XA$. Prove that

$$\operatorname{Im} T = \left\langle \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \right\rangle, \quad \operatorname{Ker} T = \langle I_2, A \rangle.$$

   [Hint: Use the standard basis $E_{11}, E_{12}, E_{21}, E_{22}$ for $M_{2\times2}(\mathbb{R})$.]

7. ($*$) Let u_1, u_2, u_3 be a basis for $\mathbb{R}^3$ and v_1, v_2, v_3 be any vectors in $\mathbb{R}^3$. If T is the linear transformation such that $T(u_i) = v_i$ for $i = 1, 2, 3$, show that $T = T_A$, where A is the 3×3 matrix

$$A = [v_1 | \, v_2 | \, v_3][u_1 | \, u_2 | \, u_3]^{-1}.$$

8. If $T : U \to V$ is a linear transformation, prove that $\operatorname{Im} T$ is a subspace of V.

9. Let $T : U \to U$ be a linear transformation. If $\operatorname{Ker} T = \{0\}$, deduce that $\operatorname{Im} T = U$.

10. Let $A \in M_{n\times n}(\mathbb{R})$ and let $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ be defined by

$$T(X) = AX.$$

   (a) Prove that X belongs to $\operatorname{Ker} T$ if and only if each column of X belongs to $N(A)$.

   (b) Prove that $\operatorname{nullity} T = n \cdot \operatorname{nullity} A$.

   (c) Prove that $\operatorname{rank} T = n \cdot \operatorname{rank} A$.

11. $(*)$ (An alternative proof of the "rank + nullity " theorem)

   Let $T : U \to V$ be a linear transformation over $\mathbb{R}$. Suppose that $u_1, \ldots, u_n$ form a basis for U, that $T(u_1), \ldots, T(u_r)$ are linearly independent and that for $i = 1, \ldots, n-r$ we have

$$T(u_{r+i}) = a_{i1}T(u_1) + \cdots + a_{ir}T(u_r),$$

   where $a_{ij} \in \mathbb{R}$ for $j = 1, \ldots, r$. Prove that the vectors

$$w_i = u_{r+i} - a_{i1}u_1 - \cdots - a_{ir}u_r, \quad 1 \le i \le n-r,$$

   form a basis for $\operatorname{Ker} T$.

12. $(*)$ Let $A \in M_{m\times n}(\mathbb{R})$. Prove that

$$\text{(a) } N(A^t) \cap C(A) = \{0\}$$

   and deduce that

$$\text{(b) } N(A^t) + C(A) = \mathbb{R}^m.$$

   [Hint: For (a), $Y \in C(A) \Rightarrow Y = AX, X \in \mathbb{R}^n; Y^tY = y_1^2 + \cdots + y_m^2$, if $Y = [y_1, \ldots, y_m]^t$.] (c) Hence deduce that the equation $A^tAX = A^tB$ is always soluble for $X \in \mathbb{R}^n$ if $B \in \mathbb{R}^m$. [Hint: Write $B = Y + Z$, where $Y \in N(A^t)$ and $Z \in C(A)$.]