($*$ indicates a challenging question.)

1. Find the coordinate vector of $v = (7, 4)^t$ relative to the basis $(3, 2)^t,\ (1, 1)^t$ of $\mathbb{R}^2$. (Ans: $(3, -2)^t$.)

2. $v_1 = (1, 1, 1)^t, v_2 = (2, 3, 2)^t,\ v_3 = (1, 5, 4)^t$ form a basis β for $\mathbb{R}^3$. Vectors $u_1 = (1, 1, 0)^t, u_2 = (1, 2, 0)^t,\ u_3 = (1, 2, 1)^t$ form a basis γ for $\mathbb{R}^3$. Find $[3v_1 + 2v_2 - v_3]_\gamma$. (Ans: $(8, -5, 3)^t$.)

3. Find bases for $R(A), C(A)$ and $N(A)$, when
(a) $A = \begin{bmatrix} 2 & 4 & 6 \\ -2 & -4 & -6 \\ 2 & 4 & 5 \end{bmatrix}$; (b) $A = \begin{bmatrix} 1 & 2 & 4 & 6 \\ 2 & 4 & 5 & -6 \end{bmatrix}$.

4. The *Lucas* numbers L_n are defined by
$$L_1 = 1,\ L_2 = 3,\ L_{n+2} = L_{n+1} + L_n, \quad n \geq 1.$$
   (a) Prove that $L_n = \alpha^n + \beta^n$, where $\alpha = (1 + \sqrt{5})/2, \beta = (1 - \sqrt{5})/2$.
   (b) Prove that $L_n = F_{n-1} + F_{n+1}$ where $F_0 = 0$. (Just observe that the RHS is a sequence $\{a_n\}$ which satisfies the Fibonacci law and that $a_n = L_n$ when $n = 1$ and $n = 2$.)

5. A sequence of real numbers $\{a_n\}$ is defined by
$$a_1 = 1,\ a_2 = 3,\ a_{n+2} = 2a_{n+1} + a_n, \quad n \geq 1.$$
Use the idea developed in lectures in the context of Fibonacci sequences to prove that
$$a_n = \frac{(1 + \sqrt{2})^n + (1 - \sqrt{2})^n}{2}.$$

6. Let V be the vector space over $\mathbb{R}$ consisting of all sequences $\{a_n\}$ of real numbers satisfying
$$a_{n+2} = 2\alpha a_{n+1} - \alpha^2 a_n, \quad n \geq 1.$$
Prove that the sequences $\{\alpha^{n-1}\}$ and $\{n\alpha^{n-1}\}$ belong to V and are linearly independent and determine down an explicit formula for a_n.

7. Let α, β, γ be distinct real numbers. Prove that the vectors $(1, \alpha, \alpha^2)$, $(1, \beta, \beta^2)$, $(1, \gamma, \gamma^2)$ are linearly independent. (Hint: show that the matrix whose rows are these vectors is non–singular.)

8. Let $U = \langle u_1,\ u_2\rangle$ and $V = \langle v_1,\ v_2\rangle$ be subspaces of W. If $u_1,\ u_2,\ v_1$ are linearly independent and $v_2 = au_1 + bu_2 + cv_1$, prove that
$$U \cap W = \langle au_1 + bu_2\rangle.$$

9. Let $u_1, \ldots, u_n$ be a linearly independent family of vectors in V and let vectors $v_1, \ldots, v_m \in V$ be defined by
$$v_i = \sum_{j=1}^{n} a_{ij} u_j, \quad 1 \leq i \leq m.$$
Prove that $v_1, \ldots, v_m$ are linearly independent if and only if the rows of the matrix $A = [a_{ij}]$ are linearly independent.

10. U and V are subspaces of $V_5(\mathbb{R})$ defined by $U = \langle u_1,\, u_2,\, u_3\rangle$, $V = \langle v_1,\, v_2,\, v_3\rangle$, where $u_1,\, u_2,\, u_3,\, v_1,\, v_2,\, v_3$ are the respective columns of the matrix A:

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 & 1 \\ 3 & 4 & 9 & 6 & 8 & 3 \\ -3 & -1 & 0 & 2 & -1 & -1 \\ -1 & -2 & -5 & -2 & -6 & -5 \\ -4 & -2 & -2 & 3 & -5 & -6 \end{bmatrix}.$$

(a) Assuming that A has reduced row–echelon form

$$B = \begin{bmatrix} 1 & 0 & -1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find bases for each of the subspaces U, V, $U+V$.

(b) Find a basis for $U \cap V$. (Hint: Use B to find vectors in $U \cap V$ and use the $\dim(U+V) + \dim(U \cap V)$ theorem for subspaces.)

11. (a) If $a_1, \ldots, a_n$ are not all zero, prove that the set

$$U = \{(x_1, \ldots, x_n)^t | a_1x_1 + \cdots + a_nx_n = 0\} \tag{1}$$

is a subspace of $\mathbb{R}^n$ with dimension equal to $n-1$.

(b) $(*)$ Prove conversely that any subspace U of $\mathbb{R}^n$ having dimension equal to $n-1$ must have the form (1).

12. $(*)$ Let $A \in M_{m\times n}(\mathbb{R})$, $B \in M_{p\times n}(\mathbb{R})$. Prove that

(a) $R(B) \subseteq R(A) \Leftrightarrow \operatorname{rank}\left[\frac{A}{B}\right] = \operatorname{rank} A$;

(b) $N(A) \subseteq N(B) \Rightarrow R(B) \subseteq R(A)$. (Hint: Use the rank + nullity theorem.)