PROBLEM SHEET 1, MP204/274, Semester 1, 1999

1. Let V be a real vector space, p a fixed vector in V. Let us define a new addition and scalar multiplication on V by the formulae

$$\begin{aligned} u \oplus v &= u + v + p, \\ t \otimes v &= tv + (t-1)p, \end{aligned}$$

for all $u,\ v \in V$, $t \in \mathbb{R}$.

Show that V is also a vector space under the new operations, with $-p$ as the zero vector and $-(v+2p)$ as the additive inverse of v.

2. Define a new addition on BR^2 by

$$(x_1, y_1) \oplus (x_2, y_2) = (\{x_1^3 + x_2^3\}^{1/3}, \{y_1^3 + y_2^3\}^{1/3}).$$

Show that with this new addition and the usual scalar multiplication, $(0,0)$ is still the additive identity and $(-x,-y)$ the additive inverse of (x,y). Also verify that all the vector space axioms hold, apart from the axiom

$$(s+t)u = su + tu.$$

3. $(*)$ If U_1 and U_2 are subspaces of a vector space V and $U_1 \cup U_2$ is also a subspace of V, prove that $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. (Hint: Prove that $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$ implies $U_1 \cup U_2$ is not closed under addition.)

4. If U and V are subspaces of a vector space W, prove that the subset of W defined by $U+V = \{u+v \mid u \in U,\ v \in V\}$ is also a subspace of W.

5. Let $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ belong to V. If $U_1 = \langle u_1, \ldots, u_m \rangle$ and $U_2 = \langle v_1, \ldots, v_n \rangle$, prove that $U_1 + U_2 = \langle u_1, \ldots, u_m, v_1, \ldots, v_n \rangle$.

6. If $u,\ v,\ w$ belong to the real vector space V, prove that

$$\langle u+v,\ v+w,\ u+w \rangle = \langle u,\ v,\ w \rangle.$$

7. U and V are subspaces of $\mathbb{R}^3$ defined by

$$U = \{(x,y,z) \mid x+y+z = 0\} \quad \text{and} \quad V = \{(x,y,z) \mid x-y-z = 0\}.$$

Find spanning families for U and V and prove that $U + V = \mathbb{R}^3$.

8. Which of the following subsets of $\mathbb{R}^2$ are subspaces of $\mathbb{R}^2$?

(a) $\{(x,y) \mid x = 3y\}$, (b) $\{(x,y) \mid x^2 = y^2\}$,
(c) $\{(x,y) \mid x + y = 1\}$, (d) $\{(x,y) \mid x \geq 0 \text{ and } y \geq 0\}$.

9. Show that the following subsets of $M_{2\times 2}(\mathbb{R})$ are also subspaces of $M_{2\times 2}(\mathbb{R})$ and find generators for these subspaces:

(a) All matrices of the form $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$, $x,\ y \in \mathbb{R}$.

(b) All matrices of the form $\begin{bmatrix} x & y \\ y & z \end{bmatrix}$, $x,\ y,\ z \in \mathbb{R}$.

10. Let $A \in M_{n\times n}(\mathbb{R})$ and let U be the subset of $M_{n\times n}(\mathbb{R})$ defined by

$$U = \{X \in M_{n\times n}(\mathbb{R}) | AX = XA\}.$$

(i) Prove that U is a subspace of $M_{n\times n}(\mathbb{R})$.

(ii) Let V be the set of matrices of the form

$$a_0 I_n + a_1 A + \cdots + a_m A^m, \quad a_0, a_1, \ldots, a_m \in \mathbb{R}.$$

Prove that V is a subspace of $M_{n\times n}(\mathbb{R})$ and that $V \subseteq U$.

(iii) Find spanning families for U and V when

(a) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, $\lambda \neq \mu$.

(Hint: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^2 = (a+d)A - (ad-bc)I_2$.)