PROBLEM SHEET 1, MP204/274, Semester 1, 1999

1. Which of the following matrices is in reduced row–echelon form?

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. [Answers: (a), (e), (g)]

2. Find reduced row–echelon forms which are row–equivalent to the following matrices:

(a) $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix}$.

[Answers:

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.]

3. Solve the following systems of linear equations by reducing the augmented matrix to reduced row–echelon form:

(a)
$$\begin{aligned} x + y + z &= 2 \\ 2x + 3y - z &= 8 \\ x - y - z &= -8 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 - x_3 + 2x_4 &= 10 \\ 3x_1 - x_2 + 7x_3 + 4x_4 &= 1 \\ -5x_1 + 3x_2 - 15x_3 - 6x_4 &= 9 \end{aligned}$$

(c)
$$\begin{aligned} 3x - y + 7z &= 0 \\ 2x - y + 4z &= \tfrac{1}{2} \\ x - y + z &= 1 \\ 6x - 4y + 10z &= 3 \end{aligned}$$

(d)
$$\begin{aligned} 2x_2 + 3x_3 - 4x_4 &= 1 \\ 2x_3 + 3x_4 &= 4 \\ 2x_1 + 2x_2 - 5x_3 + 2x_4 &= 4 \\ 2x_1 - 6x_3 + 9x_4 &= 7 \end{aligned}$$

[Answers: (a) $x = -3$, $y = \frac{19}{4}$, $z = \frac{1}{4}$; (b) inconsistent;

(c) $x = -\frac{1}{2} - 3z$, $y = -\frac{3}{2} - 2z$, with z arbitrary;

(d) $x_1 = \frac{19}{2} - 9x_4$, $x_2 = -\frac{5}{2} + \frac{17}{4}x_4$, $x_3 = 2 - \frac{3}{2}x_4$, with x_4 arbitrary.]

4. Show that the following system is consistent if and only if $c = 2a - 3b$ and solve the system in this case.

$$\begin{aligned} 2x - y + 3z &= a \\ 3x + y - 5z &= b \\ -5x - 5y + 21z &= c. \end{aligned}$$

[Answer: $x = \frac{a+b}{5} + \frac{2}{5}z$, $y = \frac{-3a+2b}{5} + \frac{19}{5}z$, with z arbitrary.]

5. For which numbers λ does the homogeneous system

$$\begin{aligned} x + (\lambda - 3)y &= 0 \\ (\lambda - 3)x + y &= 0 \end{aligned}$$

have a non–trivial solution?

[Answer: $\lambda = 2,\ 4$.]

6. Solve the homogeneous system

$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0. \end{aligned}$$

[Answer: $x_1 = -\frac{1}{4}x_3$, $x_2 = -\frac{1}{4}x_3 - x_4$, with x_3 and x_4 arbitrary.]

7. Solve the homogeneous system

$$\begin{aligned} -3x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 - 3x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 - 3x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 - 3x_4 &= 0. \end{aligned}$$

[Answer: $x_1 = x_2 = x_3 = x_4$, with x_4 arbitrary.]

8. ($*$) Let A be the coefficient matrix of the following homogeneous system of n equations in n unknowns:

$$\begin{aligned} (1-n)x_1 + x_2 + \cdots + x_n &= 0 \\ x_1 + (1-n)x_2 + \cdots + x_n &= 0 \\ \cdots &= 0 \\ x_1 + x_2 + \cdots + (1-n)x_n &= 0. \end{aligned}$$

Find the reduced row–echelon form of A and hence, or otherwise, prove that the solution of the above system is $x_1 = x_2 = \cdots = x_n$, with x_n arbitrary.

9. Let A be $n \times n$.

   (i) If $A^2 = 0$, prove that A is singular.

   (ii) If $A^2 = A$ and $A \neq I_n$, prove that A is singular.

10. Determine explicitly the following products of 3×3 elementary row matrices.

(i) $E_{12}E_{23}$ (ii) $E_1(5)E_{12}$ (iii) $E_{12}(3)E_{21}(-3)$ (iv) $(E_1(100))^{-1}$
(v) E_{12}^{-1} (vi) $(E_{12}(7))^{-1}$ (vii) $(E_{12}(7)E_{31}(1))^{-1}$.

[Answers: (i) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} -8 & 3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} \frac{1}{100} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & -7 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (vii) $\begin{bmatrix} 1 & -7 & 0 \\ 0 & 1 & 0 \\ -1 & 7 & 1 \end{bmatrix}$.]

11. Let A be the following product of 4×4 elementary row matrices:

$$A = E_3(2)E_{14}E_{42}(3).$$

Find A and A^{-1} explicitly.

[Answers: $A = \begin{bmatrix} 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 1 & -3 & 0 & 0 \end{bmatrix}$.]

12. Let A be $m \times m$. If A is row–equivalent to a matrix having a zero row, prove that A is singular.

13. Determine which of the following matrices are non–singular and find the inverse, where possible.

(a) $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 2 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 6 & -3 \\ 0 & 0 & 7 \\ 0 & 0 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix}$.

[Answers: (a) $\begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 1 & -1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{1}{2} & 2 & 1 \\ 0 & 0 & 1 \\ \frac{1}{2} & -1 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix}$.]

14. (∗) (i) If A is $m \times n$ and B is $n \times m$, prove that AB is singular if $m > n$.

(ii) Let A and B be $m \times m$. If A or B is singular, prove that AB is also singular.

15. (∗) Let A be $m \times m$.

(i) If for all $X \in \mathbb{R}^m$, $AX = 0 \Rightarrow X = 0$, prove that A is non–singular.

(ii) If B is $m \times m$ and $BA = I_m$, deduce that $AB = I_m$. (Hint: use (i).)

(iii) If A is non–singular, prove that $\mathrm{rref}(A) = I_m$.

16. (∗) Let B be an $n \times n$ *skew–symmetric* matrix, ie. $B^t = -B$. Prove that $A = I_n - B$ is non–singular. (Hint: Let $X \in \mathbb{R}^n$, assume $AX = 0$, deduce $X^tX = X^tBX$ and deduce $X = 0$ by using the fact that X^tBX is 1×1 and hence equal to its transpose.)