NON–SINGULAR MATRICES

DEFINITION. (Non–singular matrix) An $n \times n$ A is called *non–singular* or *invertible* if there exists an $n \times n$ matrix B such that

$$AB = I_n = BA.$$

Any matrix B with the above property is called an *inverse* of A. If A does not have an inverse, A is called *singular*.

THEOREM. (Inverses are unique) If A has inverses B and C, then $B = C$.

If A has an inverse, it is denoted by A^{-1}. So

$$AA^{-1} = I_n = A^{-1}A.$$

Also if A is non–singular, then A^{-1} is also non–singular and

$$(A^{-1})^{-1} = A.$$

THEOREM. If A and B are non–singular matrices of the same size, then so is AB. Moreover

$$(AB)^{-1} = B^{-1}A^{-1}.$$

The above result generalizes to a product of m non–singular matrices: If $A_1, \ldots, A_m$ are non–singular $n \times n$ matrices, then the product $A_1 \ldots A_m$ is also non–singular. Moreover

$$(A_1 \ldots A_m)^{-1} = A_m^{-1} \ldots A_1^{-1}.$$

(Thus the inverse of the product equals the product of the inverses *in the reverse order*.)

THEOREM. Let $A = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ and $\Delta = ad - bc \neq 0$. Then A is non–singular. Also

$$A^{-1} = \Delta^{-1} \left[\begin{array}{rr} d & -b \\ -c & a \end{array}\right].$$

REMARK. The expression $ad - bc$ is called the *determinant* of A and is denoted by the symbols $\det A$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

THEOREM. If the coefficient matrix A of a system of n equations in n unknowns is non–singular, then the system $AX = B$ has the unique solution $X = A^{-1}B$.

Proof. Assume that A^{-1} exists.

(Uniqueness.) Assume that $AX = B$. Then

$$\begin{aligned} (A^{-1}A)X &= A^{-1}B, \\ I_nX &= A^{-1}B, \\ X &= A^{-1}B. \end{aligned}$$

(Existence.) Let $X = A^{-1}B$. Then

$$AX = A(A^{-1}B) = (AA^{-1})B = I_nB = B.$$

COROLLARY. If A is an $n \times n$ non–singular matrix, then the homogeneous system $AX = 0$ has only the trivial solution $X = 0$. Hence if the system $AX = 0$ has a non–trivial solution, A is singular.

EXAMPLE.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 4 & 7 \end{bmatrix}$$

is singular. For it can be verified that A has reduced row–echelon form

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and consequently $AX = 0$ has a non–trivial solution $x = -1$, $y = -1$, $z = 1$.

REMARK. More generally, if A is row–equivalent to a matrix containing a zero row, then A is singular. For then the homogeneous system $AX = 0$ has a non–trivial solution.

THEOREM (Cramer's rule for 2 equations in 2 unknowns.) The system

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

has a unique solution if $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, namely

$$x = \frac{\Delta_1}{\Delta}, \qquad y = \frac{\Delta_2}{\Delta},$$

where

$$\Delta_1 = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \quad \text{and} \quad \Delta_2 = \begin{vmatrix} a & e \\ c & f \end{vmatrix}.$$

PROOF. Suppose $\Delta \neq 0$. Then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has inverse

$$A^{-1} = \Delta^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and we know that the system

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

has the unique solution

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} e \\ f \end{bmatrix} &= \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} de - bf \\ -ce + af \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} \\ &= = \begin{bmatrix} \Delta_1/\Delta \\ \Delta_2/\Delta \end{bmatrix}. \end{aligned}$$

Hence $x = \Delta_1/\Delta$, $y = \Delta_2/\Delta$.

COROLLARY. The homogeneous system

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

has only the trivial solution if $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

EXAMPLE. The system

$$\begin{aligned} 7x + 8y &= 100 \\ 2x - 9y &= 10 \end{aligned}$$

has the unique solution
$x = \Delta_1/\Delta$, $y = \Delta_2/\Delta$, where

$$\begin{aligned} \Delta &= \begin{vmatrix} 7 & 8 \\ 2 & -9 \end{vmatrix} = -79, \\ \Delta_1 &= \begin{vmatrix} 100 & 8 \\ 10 & -9 \end{vmatrix} = -980, \\ \Delta_2 &= \begin{vmatrix} 7 & 100 \\ 2 & 10 \end{vmatrix} = -130. \end{aligned}$$

So $x = \frac{980}{79}$ and $y = \frac{130}{79}$.

ELEMENTARY ROW MATRICES.

An important class of non–singular matrices is that of the *elementary row matrices*.

DEFINITION. (Elementary row matrices) There are three types, E_{ij}, $E_i(t)\,, E_{ij}(t)$.

E_{ij}, $(i \neq j)$ is obtained from the identity matrix I_n by interchanging rows i and j.

$E_i(t)$, $(t \neq 0)$ is obtained by multiplying the i–th row of I_n by t.

$E_{ij}(t)$, $(i \neq j)$ is obtained from I_n by adding t times the j–th row of I_n to the i–th row.

EXAMPLE. ($n = 3$.)

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_{23}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The elementary row matrices have the following distinguishing property:

THEOREM. If a matrix A is pre–multiplied by an elementary row–matrix, the resulting matrix is the one obtained by performing the corresponding elementary row–operation on A.

EXAMPLE.

$$E_{23} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} a & b \\ e & f \\ c & d \end{bmatrix}.$$

THEOREM. The three types of elementary row–matrices are non–singular. In fact

$$\begin{aligned} E_{ij}E_{ij} &= I_n \\ E_i(t)E_i(t^{-1}) &= I_n = E_i(t^{-1})E_i(t) \quad \text{if } t \neq 0 \\ E_{ij}(t)E_{ij}(-t) &= I_n = E_{ij}(-t)E_{ij}(t). \end{aligned}$$

EXAMPLE. Find the 3×3 matrix $A = E_3(5)E_{23}(2)E_{12}$ explicitly. Also find A^{-1}.

SOLUTION.

$$\begin{aligned} A &= E_3(5)E_{23}(2)\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= E_3(5)\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}. \end{aligned}$$

To find A^{-1}, we have

$$\begin{aligned}
A^{-1} &= (E_3(5)E_{23}(2)E_{12})^{-1} \\
&= E_{12}^{-1}\,(E_{23}(2))^{-1}\,(E_3(5))^{-1} \\
&= E_{12}E_{23}(-2)E_3(5^{-1}) \\
&= E_{12}E_{23}(-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \\
&= E_{12}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\frac{2}{5} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}.
\end{aligned}$$

THEOREM. Let A be $n \times n$ and suppose that A is row–equivalent to I_n. Then A is non–singular and A^{-1} can be found by performing the same sequence of elementary row operations on I_n as were used to convert A to I_n.

PROOF. Suppose that $E_r \ldots E_1 A = I_n$. In other words $BA = I_n$, where $B = E_r \ldots E_1$ is non–singular. Then $B^{-1}(BA) = B^{-1}I_n$ and so $A = B^{-1}$, which is non–singular.

Also
$A^{-1} = \left(B^{-1}\right)^{-1} = B = E_r\left((\ldots(E_1 I_n)\ldots\right)$,
which shows that A^{-1} is obtained from I_n by performing the same sequence of elementary row operations as were used to convert A to I_n.

EXAMPLE. Show that $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ is non–singular, find A^{-1} and express A as a product of elementary row matrices.

SOLUTION. We form the *partitioned* matrix $[A|I_2]$ which consists of A followed by I_2. Then any sequence of elementary row operations which reduces A to I_2 will reduce I_2 to A^{-1}. Here

$$[A|I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - R_1 \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right]$$

$$R_2 \to (-1)R_2 \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$R_1 \to R_1 - 2R_2 \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right].$$

Hence A is row–equivalent to I_2 and A is non–singular. Also

$$A^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

We also observe that

$$E_{12}(-2)E_2(-1)E_{21}(-1)A = I_2.$$

Hence

$$\begin{aligned} A^{-1} &= E_{12}(-2)E_2(-1)E_{21}(-1) \\ A &= E_{21}(1)E_2(-1)E_{12}(2). \end{aligned}$$

The next result is occasionally useful for proving the non–singularity of certain types of matrices.

THEOREM. Let A be an $n \times n$ matrix with the property that the homogeneous system $AX = 0$ has only the trivial solution. Then A is non–singular. Equivalently, if A is singular, then the homogeneous system $AX = 0$ has a non–trivial solution.

PROOF. If A is $n \times n$ and the homogeneous system $AX = 0$ has only the trivial solution, then it follows that the reduced row–echelon form B of A cannot have zero rows and must therefore be I_n. Hence A is non–singular.

COROLLARY. Suppose that A and B are $n \times n$ and $AB = I_n$. Then $BA = I_n$.

PROOF. Let $AB = I_n$, where A and B are $n \times n$. We first show that B is non–singular. Assume $BX = 0$. Then $A(BX) = A0 = 0$, so $(AB)X = 0$, $I_nX = 0$ and hence $X = 0$.

Then from $AB = I_n$ we deduce $(AB)B^{-1} = I_nB^{-1}$ and hence $A = B^{-1}$.

The equation $BB^{-1} = I_n$ then gives $BA = I_n$.