Generalised Eigenspaces and the block upper triangular form

In pp. 300–304 of his book *Linear Algebra*, M. O’Nan gives a simple algorithm for finding a block upper triangular form similar to a given matrix whose characteristic polynomial splits completely. O’Nan’s justification of the algorithm is based on a modification of the standard Schur upper triangulation method and is somewhat messy.

The point of this note is to justify the simpler algorithm in a more elegant and direct manner.
THEOREM. Let $\lambda$ be an eigenvalue of $A \in M_{n \times n}(\mathbb{R})$, $B = A - \lambda I_n$ and $a_A(\lambda)$ be the algebraic multiplicity of $\lambda$. Then for $b \geq 1$,
\[
\text{nullity } B^b \leq a_A(\lambda).
\]

PROOF. The proof is an extension of the standard proof (using determinants) of the inequality $g_A(\lambda) \leq a_A(\lambda)$, where $g_A(\lambda) = \text{nullity } (B)$.

Start with a basis $X_1, \ldots, X_r$ of $N(B)$. Extend this to a basis of $N(B^2)$:
\[X_1, \ldots, X_r, Y_1, \ldots, Y_s.\]
Keep extending until we reach a basis of $N(B^b)$.

Then, in addition to the equations
\[AX_1 = \lambda X_1, \ldots, AX_r = \lambda X_r,\]
we have $B^2 Y_1 = 0$, so $BY_1 \in N(B)$ and hence
\[ BY_1 = b_{11}X_1 + \cdots + b_{r1}X_r \] and hence
\[ AX_1 = b_{11}X_1 + \cdots + b_{r1}X_r + \lambda Y_1 \] etc.

Now complete the above basis to a basis of \( \mathbb{R}^n \) and let \( P \) be the matrix whose columns are these respective vectors. Then
\[
P^{-1}AP = \begin{bmatrix} D & B_1 \\ 0 & B_2 \end{bmatrix},
\]
where \( D \) is an upper triangular matrix with \( \lambda' \)s on the diagonal and of size \( t \times t \), where \( t = \text{nullity } B^b \).

Hence \( ch_A(x) = (x - \lambda)^t g(x) \), where \( g(x) = ch_{B_2}(x) \) and consequently \( t \leq a_A(\lambda) \).
COROLLARY. \( \dim G_A(\lambda) = a_A(\lambda) \), where \( G_A(\lambda) = N(B^{a_A(\lambda)}) \) is the generalised eigenspace corresponding to eigenvalue \( \lambda \) and \( B = A - \lambda I_n \).

PROOF. Let \( ch_A(x) = (x - c_1)^{a_1} \cdots (x - c_t)^{a_t} \), so that \( a_i = a_A(c_i) \). Then the Cayley–Hamilton theorem gives

\[
0 = ch_A(A) = (A - c_1 I_n)^{a_1} \cdots (A - c_t I_n)^{a_t},
\]
or

\[
0 = B_1^{a_1} \cdots B_t^{a_t}, \text{ say.}
\]

Now take nullities of both sides:

\[
n = \text{nullity } 0 \leq \sum_{i=1}^{t} \text{nullity } B_i^{a_i} \leq \sum_{i=1}^{t} a_i = n.
\]

Hence we have equality at all places and \( \text{nullity } B_i^{a_i} = a_i \) for \( 1 \leq i \leq t \).
THEOREM. Let 
\[ ch_A(x) = (x - c_1)^{a_1} \cdots (x - c_t)^{a_t}. \]
Then the generalised eigenspaces \( G_A(c_1), \ldots, G_A(c_t) \) are independent.

PROOF. Assume \( X_1 + \cdots + X_t = 0 \), where \( X_i \in G_A(c_i), \ 1 \leq i \leq t \).
Without loss of generality, we show \( X_1 = 0 \).
Let \( B = (A - c_2 I_n)^{a_2} \cdots (A - c_t I_n)^{a_t} \).
Then \( B(X_1 + \cdots + X_t) = B0 = 0 \).
Hence \( BX_1 = 0 \), as \( BX_2 = 0, \ldots, BX_t = 0 \).
Now \( \gcd \{ (x - c_1)^{a_1}, (x - c_2)^{a_2} \cdots (x - c_t)^{a_t} \} = 1 \),
so there exist polynomials \( u(x), v(x) \) such that
\[ u(x)(x - c_1)^{a_1} + v(x)(x - c_2)^{a_2} \cdots (x - c_t)^{a_t} = 1. \]
Hence
\[ u(A)(A - c_1 I_n)^{a_1} + v(A)B = I_n. \]
Finally
\[ u(A)(A - c_1 I_n)^{a_1}X_1 + v(A)BX_1 = I_nX_1 = X_1, \]
so \( u(A)0 + v(A)0 = X_1 \) and consequently \( X_1 = 0 \).