Generalised Eigenspaces and the block upper triangular form

In pp. 300–304 of his book *Linear Algebra*, M. O'Nan gives a simple algorithm for finding a block upper triangular form similar to a given matrix whose characteristic polynomial splits completely. O'Nan's justification of the algorithm is based on a modification of the standard Schur upper triangulation method and is somewhat messy.

The point of this note is to justify the simpler algorithm in a more elegant and direct manner.

THEOREM. Let λ be an eigenvalue of $A \in M_{n\times n}(\mathbb{R})$, $B = A - \lambda I_n$ and $a_A(\lambda)$ be the algebraic multiplicity of λ. Then for $b \geq 1$,

$$\text{nullity}\, B^b \leq a_A(\lambda).$$

PROOF. The proof is an extension of the standard proof (using determinants) of the inequality $g_A(\lambda) \leq a_A(\lambda)$, where $g_A(\lambda) = \text{nullity}\,(B)$.

Start with a basis $X_1, \ldots, X_r$ of $N(B)$. Extend this to a basis of $N(B^2)$:

$X_1, \ldots, X_r, Y_1, \ldots, Y_s.$

Keep extending until we reach a basis of $N(B^b)$.

Then, in addition to the equations

$AX_1 = \lambda X_1, \ldots, AX_r = \lambda X_r,$

we have $B^2 Y_1 = 0$, so $BY_1 \in N(B)$ and hence

$BY_1 = b_{11}X_1 + \cdots + b_{r1}X_r$ and hence

$AX_1 = b_{11}X_1 + \cdots + b_{r1}X_r + \lambda Y_1$ etc.

Now complete the above basis to a basis of $\mathbb{R}^n$ and let P be the matrix whose columns are these respective vectors. Then

$$P^{-1}AP = \left[\begin{array}{c|c} D & B_1 \\ \hline 0 & B_2 \end{array}\right],$$

where D is an upper triangular matrix with $\lambda's$ on the diagonal and of size $t \times t$, where $t =$ nullity B^b.

Hence $ch_A(x) = (x - \lambda)^t g(x)$, where $g(x) = ch_{B_2}(x)$ and consequently $t \leq a_A(\lambda)$.

COROLLARY. $\dim G_A(\lambda) = a_A(\lambda)$, where $G_A(\lambda) = N(B^{a_A(\lambda)})$ is the generalised eigenspace corresponding to eigenvalue λ and $B = A - \lambda I_n$.

PROOF. Let $ch_A(x) = (x - c_1)^{a_1} \cdots (x - c_t)^{a_t}$, so that $a_i = a_A(c_i)$. Then the Cayley–Hamilton theorem gives

$0 = ch_A(A) = (A - c_1 I_n)^{a_1} \cdots (A - c_t I_n)^{a_t}$, or

$0 = B_1^{a_1} \cdots B_t^{a_t}$, say.

Now take nullities of both sides:

$$n = \text{nullity}\, 0 \leq \sum_{i=1}^{t} \text{nullity}\, B_i^{a_i} \leq \sum_{i=1}^{t} a_i = n.$$

Hence we have equality at all places and nullity $B_i^{a_i} = a_i$ for $1 \leq i \leq t$.

THEOREM. Let
$ch_A(x) = (x - c_1)^{a_1} \cdots (x - c_t)^{a_t}$. Then the generalised eigenspaces $G_A(c_1), \ldots, G_A(c_t)$ are independent.

PROOF. Assume $X_1 + \cdots + X_t = 0$, where $X_i \in G_A(c_i),\ 1 \leq i \leq t$.

Without loss of generality, we show $X_1 = 0$.

Let $B = (A - c_2 I_n)^{a_2} \cdots (A - c_t I_n)^{a_t}$.

Then $B(X_1 + \cdots + X_t) = B0 = 0$.

Hence $BX_1 = 0$, as $BX_2 = 0, \ldots, BX_t = 0$.

Now
$\gcd\{(x - c_1)^{a_1}, (x - c_2)^{a_2} \cdots (x - c_t)^{a_t}\} = 1$,

so there exist polynomials $u(x), v(x)$ such that

$u(x)(x - c_1)^{a_1} + v(x)(x - c_2)^{a_2} \cdots (x - c_t)^{a_t} = 1$.
Hence

$u(A)(A - c_1 I_n)^{a_1} + v(A)B = I_n$.

Finally
$u(A)(A - c_1 I_n)^{a_1} X_1 + v(A)BX_1 = I_n X_1 = X_1$,

so $u(A)0 + v(A)0 = X_1$ and consequently
$X_1 = 0$.