

1. LINEAR EQUATIONS

A *linear equation* in n unknowns x_1, x_2, \dots, x_n is an equation of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b,$$

where a_1, a_2, \dots, a_n, b are given real numbers.

For example, with x and y instead of x_1 and x_2 , the linear equation $2x + 3y = 6$ describes the line passing through the points $(3, 0)$ and $(0, 2)$.

Similarly, with x, y and z instead of x_1, x_2 and x_3 , the linear equation $2x + 3y + 4z = 12$ describes the plane passing through the points $(6, 0, 0), (0, 4, 0), (0, 0, 3)$.

A *system* of m linear equations in n unknowns x_1, x_2, \dots, x_n is a family of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

We wish to determine if such a system has a solution, that is to find out if there exist numbers x_1, x_2, \dots, x_n which satisfy each of the equations simultaneously. We say that the system is *consistent* if it has a solution. Otherwise the system is called *inconsistent*.

Note that the above system can be written concisely as

$$\sum_{j=1}^n a_{ij}x_j = b_i, \quad i = 1, 2, \dots, m.$$

The matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called the *coefficient matrix* of the system, while the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

is called the *augmented matrix* of the system.

Geometrically, solving a system of linear equations in two (or three) unknowns is equivalent to determining whether or not a family of lines (or planes) has a common point of intersection.

EXAMPLE. Solve the equation

$$2x + 3y = 6.$$

SOLUTION. The equation $2x + 3y = 6$ is equivalent to $2x = 6 - 3y$ or $x = 3 - \frac{3}{2}y$, where y is arbitrary. So there are infinitely many solutions.

EXAMPLE. Solve the system

$$\begin{aligned}x + y + z &= 1 \\x - y + z &= 0.\end{aligned}$$

SOLUTION. We subtract the second equation from the first, to get $2y = 1$ and $y = \frac{1}{2}$. Then $x = y - z = \frac{1}{2} - z$, where z is arbitrary. Again there are infinitely many solutions.

EXAMPLE. Find a polynomial of the form $y = a_0 + a_1x + a_2x^2 + a_3x^3$ whose graph passes through the points $(-3, -2)$, $(-1, 2)$, $(1, 5)$, $(2, 1)$.

SOLUTION. When x has the values $-3, -1, 1, 2$, then y takes corresponding values $-2, 2, 5, 1$ and we get four equations in the unknowns a_0, a_1, a_2, a_3 :

$$a_0 - 3a_1 + 9a_2 - 27a_3 = -2$$

$$a_0 - a_1 + a_2 - a_3 = 2$$

$$a_0 + a_1 + a_2 + a_3 = 5$$

$$a_0 + 2a_1 + 4a_2 + 8a_3 = 1.$$

This system has the unique solution $a_0 = 93/20$, $a_1 = 221/120$, $a_2 = -23/20$, $a_3 = -41/120$. So the required polynomial is

$$y = \frac{93}{20} + \frac{221}{120}x - \frac{23}{20}x^2 - \frac{41}{120}x^3.$$

Solving a system consisting of a single linear equation is easy. However if we are dealing with two or more equations, it is desirable to have a systematic method of determining if the system is consistent and to find all solutions.

Instead of restricting ourselves to linear equations with rational or real coefficients, our theory goes over to the more general case where the coefficients belong to an arbitrary *field*. A *field* F is a set F which possesses operations of *addition* and *multiplication* which satisfy the familiar rules of rational arithmetic. The complex numbers \mathbb{C} and \mathbb{Z}_p , the congruence classes mod p , where p is a prime, are useful fields.

2. THE GAUSS JORDAN ALGORITHM

We show how to solve any system of linear equations over an arbitrary field, using the *GAUSS–JORDAN* algorithm. We first need to define some terms.

DEFINITION. (Row–echelon form) A matrix is in *row–echelon form* if

- (i) all zero rows (if any) are at the bottom of the matrix and
- (ii) if two successive rows are non–zero, the second row starts with more zeros than the first (moving from left to right).

For example, the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row–echelon form, whereas the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is not in row–echelon form.

The *zero* matrix of any size is always in row–echelon form.

DEFINITION. (Reduced row–echelon form) A matrix is in *reduced row–echelon form* if

1. it is in row–echelon form,
2. the leading (leftmost non–zero) entry in each non–zero row is 1,
3. all other elements of the column in which the leading entry 1 occurs are zeros.

For example the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are in reduced row–echelon form, whereas the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are not in reduced row–echelon form, but are in row–echelon form.

The *zero* matrix of any size is always in reduced row–echelon form.

NOTATION. If a matrix A is in reduced row–echelon form, it is useful to denote the column numbers in which the leading entries 1 occur, by c_1, c_2, \dots, c_r , with the remaining column numbers being denoted by c_{r+1}, \dots, c_n , where r is the number of non–zero rows of A . For example, in the 4×6 matrix above, we have $r = 3$, $c_1 = 2$, $c_2 = 4$, $c_3 = 5$, $c_4 = 1$, $c_5 = 3$, $c_6 = 6$.

The following operations are the ones used on systems of linear equations and do not change the solution set.

DEFINITION. (Elementary row operations)
There are three types of *elementary row operations* that can be performed on matrices:

1. Interchanging rows i and j :

$$R_i \leftrightarrow R_j$$

2. Multiplying row i by a non-zero number t :

$$R_i \rightarrow tR_i$$

3. Adding t times row i to row j :

$$R_j \rightarrow R_j + tR_i$$

DEFINITION (Row equivalence) Matrix A is *row-equivalent* to matrix B if B is obtained from A by a sequence of elementary row operations.

EXAMPLE. Working from left to right,

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \quad R_2 \rightarrow R_2 + 2R_3 \quad \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 5 \\ 1 & -1 & 2 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 2 \\ 4 & -1 & 5 \end{bmatrix} \quad R_1 \rightarrow 2R_1$$

$$\begin{bmatrix} 2 & 4 & 0 \\ 1 & -1 & 2 \\ 4 & -1 & 5 \end{bmatrix} = B.$$

Thus A is row-equivalent to B . Clearly B is also row-equivalent to A , by performing the inverse row-operations

$$R_1 \rightarrow \frac{1}{2}R_1, R_2 \leftrightarrow R_3, R_2 \rightarrow R_2 - 2R_3 \text{ on } B.$$

It is not difficult to prove that if A and B are row-equivalent augmented matrices of two systems of linear equations, then the two systems have the same solution sets – a solution of the one system is a solution of the other. For example the systems whose augmented matrices are A and B in the above example are respectively

$$\begin{cases} x + 2y = 0 \\ 2x + y = 1 \\ x - y = 2 \end{cases} \quad \text{and} \quad \begin{cases} 2x + 4y = 0 \\ x - y = 2 \\ 4x - y = 5 \end{cases}$$

and these systems have precisely the same solutions.

THE GAUSS–JORDAN ALGORITHM.

This is a process which starts with a given matrix A and produces a matrix B in reduced row–echelon form, which is row–equivalent to A . If A is the augmented matrix of a system of linear equations, then B will be a much simpler matrix than A from which the consistency or inconsistency of the corresponding system is immediately apparent and in fact the complete solution of the system can be read off.

STEP 1.

Find the first non-zero column moving from left to right, (column c_1) and select a non-zero entry from this column. By interchanging rows, if necessary, ensure that the first entry in this column is non-zero. Multiply row 1 by the multiplicative inverse of a_{1c_1} thereby converting a_{1c_1} to 1. For each non-zero element a_{ic_1} , $i > 1$, (if any) in column c_1 , add $-a_{ic_1}$ times row 1 to row i , thereby ensuring that all elements in column c_1 , apart from the first, are zero.

STEP 2. If the matrix obtained at Step 1 has its 2nd, ..., m th rows all zero, the matrix is in reduced row–echelon form. Otherwise suppose that the first column which has a non–zero element in the rows below the first is column c_2 . Then $c_1 < c_2$. By interchanging rows below the first, if necessary, ensure that a_{2c_2} is non–zero. Then convert a_{2c_2} to 1 and by adding suitable multiples of row 2 to the remaining rows, where necessary, ensure that all remaining elements in column c_2 are zero.

The process is repeated and will eventually stop after r steps, either because we run out of rows, or because we run out of non–zero columns. In general, the final matrix will be in reduced row–echelon form and will have r non–zero rows, with leading entries 1 in columns c_1, \dots, c_r , respectively.

EXAMPLE

$$\begin{bmatrix} 0 & 0 & 4 & 0 \\ 2 & 2 & -2 & 5 \\ 5 & 5 & -1 & 5 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 2 & 2 & -2 & 5 \\ 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \end{bmatrix}$$

$$R_1 \rightarrow \frac{1}{2}R_1 \begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 5R_1 \begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & -\frac{15}{2} \end{bmatrix}$$

$$R_2 \rightarrow \frac{1}{4}R_2 \begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -\frac{15}{2} \end{bmatrix}$$

$$\begin{cases} R_1 \rightarrow R_1 + R_2 \\ R_3 \rightarrow R_3 - 4R_2 \end{cases} \begin{bmatrix} 1 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{15}{2} \end{bmatrix}$$

$$R_3 \rightarrow \frac{-2}{15}R_3 \begin{bmatrix} 1 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - \frac{5}{2}R_3 \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The last matrix is in reduced row–echelon form.

REMARK. It is possible to show that a given matrix over an arbitrary field is row–equivalent to *precisely one* matrix which is in reduced row–echelon form.

3. SYSTEMATIC SOLUTION OF LINEAR SYSTEMS

Suppose a system of m linear equations in n unknowns x_1, \dots, x_n has augmented matrix A and that A is row-equivalent to a matrix B which is in reduced row-echelon form, via the Gauss-Jordan algorithm. Then A and B are $m \times (n + 1)$. Suppose that B has r non-zero rows and that the leading entry 1 in row i occurs in column number c_i , for $1 \leq i \leq r$. Then

$$1 \leq c_1 < c_2 < \dots < c_r \leq n + 1.$$

Also assume that the remaining column numbers are c_{r+1}, \dots, c_{n+1} , where

$$1 \leq c_{r+1} < c_{r+2} < \dots < c_{n+1} \leq n + 1.$$

Case 1: $c_r = n + 1$. The system is inconsistent. For the last non-zero row of B is $[0, 0, \dots, 1]$ and the corresponding equation is

$$0x_1 + 0x_2 + \dots + 0x_n = 1,$$

which has no solutions. Consequently the original system has no solutions.

Case 2: $c_r \leq n$. The system of equations corresponding to the non-zero rows of B is consistent. First notice that $r \leq n$ here.

If $r = n$, then $c_1 = 1, c_2 = 2, \dots, c_n = n$ and

$$B = \begin{bmatrix} 1 & 0 & \dots & 0 & d_1 \\ 0 & 1 & \dots & 0 & d_2 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & d_n \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

There is a unique solution

$$x_1 = d_1, x_2 = d_2, \dots, x_n = d_n.$$

If $r < n$, there will be more than one solution (infinitely many if the field is infinite). For all solutions are obtained by taking the unknowns x_{c_1}, \dots, x_{c_r} as *dependent* unknowns and using the r equations corresponding to the non-zero rows of B to express these unknowns in terms of the remaining *independent* unknowns $x_{c_{r+1}}, \dots, x_{c_n}$, which can take on arbitrary values:

$$\begin{aligned} x_{c_1} &= b_{1n+1} - b_{1c_{r+1}}x_{c_{r+1}} - \dots - b_{1c_n}x_{c_n} \\ &\vdots \\ x_{c_r} &= b_{rn+1} - b_{rc_{r+1}}x_{c_{r+1}} - \dots - b_{rc_n}x_{c_n}. \end{aligned}$$

In particular, taking $x_{c_{r+1}} = 0, \dots, x_{c_{n-1}} = 0$ and $x_{c_n} = 0, 1$ respectively, produces at least two solutions.

EXAMPLE. Solve the system

$$\begin{aligned}x + y &= 0 \\x - y &= 1 \\4x + 2y &= 1.\end{aligned}$$

SOLUTION. The augmented matrix of the system is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

which is row equivalent to

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

We read off the unique solution
 $x = \frac{1}{2}$, $y = -\frac{1}{2}$.

(Here $n = 2$, $r = 2$, $c_1 = 1$, $c_2 = 2$. Also $c_r = c_2 = 2 < 3 = n + 1$ and $r = n$.)

EXAMPLE. Solve the system

$$\begin{aligned}2x_1 + 2x_2 - 2x_3 &= 5 \\7x_1 + 7x_2 + x_3 &= 10 \\5x_1 + 5x_2 - x_3 &= 5.\end{aligned}$$

SOLUTION. The augmented matrix is

$$A = \begin{bmatrix} 2 & 2 & -2 & 5 \\ 7 & 7 & 1 & 10 \\ 5 & 5 & -1 & 5 \end{bmatrix}$$

which is row equivalent to

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We read off inconsistency for the original system.

(Here $n = 3$, $r = 3$, $c_1 = 1$, $c_2 = 3$. Also $c_r = c_3 = 4 = n + 1$.)

EXAMPLE, Solve the system

$$\begin{aligned}x_1 - x_2 + x_3 &= 1 \\x_1 + x_2 - x_3 &= 2.\end{aligned}$$

Solution. The augmented matrix is

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

which is row equivalent to

$$B = \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & -1 & \frac{1}{2} \end{bmatrix}.$$

The complete solution is $x_1 = \frac{3}{2}$, $x_2 = \frac{1}{2} + x_3$, with x_3 arbitrary.

(Here $n = 3$, $r = 2$, $c_1 = 1$, $c_2 = 2$. Also $c_r = c_2 = 2 < 4 = n + 1$ and $r < n$.)

EXAMPLE. Solve the system

$$\begin{aligned}6x_3 + 2x_4 - 4x_5 - 8x_6 &= 8 \\3x_3 + x_4 - 2x_5 - 4x_6 &= 4 \\2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 &= 2 \\6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 &= 1.\end{aligned}$$

SOLUTION. The augmented matrix is

$$A = \begin{bmatrix} 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{bmatrix}$$

which is row equivalent to

$$B = \begin{bmatrix} 1 & -\frac{3}{2} & 0 & \frac{11}{6} & -\frac{19}{6} & 0 & \frac{1}{24} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & 0 & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The complete solution is

$$x_1 = \frac{1}{24} + \frac{3}{2}x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5,$$

$$x_3 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5,$$

$$x_6 = \frac{1}{4},$$

with x_2, x_4, x_5 arbitrary.

(Here $n = 6, r = 3, c_1 = 1, c_2 = 3, c_3 = 6;$
 $c_r = c_3 = 6 < 7 = n + 1; r < n.$)

4. HOMOGENEOUS SYSTEMS

A system of homogeneous linear equations is a system of the form

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0.\end{aligned}$$

Such a system is always consistent as $x_1 = 0, \dots, x_n = 0$ is a solution. This solution is called the *trivial* solution. Any other solution is called a *non-trivial* solution.

For example the homogeneous system

$$\begin{aligned}x - y &= 0 \\x + y &= 0\end{aligned}$$

has only the trivial solution, whereas the homogeneous system

$$\begin{aligned}x - y + z &= 0 \\x + y + z &= 0\end{aligned}$$

has the complete solution $x = -z$, $y = 0$, z arbitrary. In particular, taking $z = 1$ gives the non-trivial solution $x = -1$, $y = 0$, $z = 1$.

There is a simple but fundamental theorem concerning homogeneous systems.

THEOREM. A homogeneous system of m linear equations in n unknowns always has a non-trivial solution if $m < n$.

PROOF. Suppose that $m < n$ and that the coefficient matrix of the system is row-equivalent to B , a matrix in reduced row-echelon form. Let r be the number of non-zero rows in B . Then $r \leq m < n$ and hence $n - r > 0$ and so the number $n - r$ of arbitrary unknowns is in fact positive. Taking one of these unknowns to be 1 gives a non-trivial solution.