INTERNAL STUDENTS ONLY

THE UNIVERSITY OF QUEENSLAND

First Semester Examination, June 1999.

MP204/MP274

LINEAR ALGEBRA II/IIH

(UNIT COURSES)

Time: **TWO HOURS** for working
Ten minutes for perusal before examination begins.

Attempt **five (5) questions only**.
Question 8 is compulsory for MP274 students.
All questions carry the same number of marks.
Candidates must not remove this paper from the examination room.
Pocket calculators allowed.

1. (a) Define the terms *null–space* $N(A)$ *of* A, *column–space* $C(A)$ *of* A, *rank* A, *nullity* A, where $A \in M_{m\times n}(\mathbb{R})$.

   (b) For the following matrix A, find the reduced row–echelon form of A, a basis for $N(A)$ and $\operatorname{rank} A$ and $\operatorname{nullity} A$:

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -2 & 1 & 3 \\ -2 & 2 & -7 & 3 \end{bmatrix}.$$

   Also explain why $(1,5,1)^t$ and $(0,3,3)^t$ form a basis for $C(A)$.

2. (a) Let $v_1, \ldots, v_n$ belong to a vector space V. Explain what is meant by the statements (i) $v_1, \ldots, v_n$ *are linearly dependent*, (ii) $v_1, \ldots, v_n$ *form a basis for* V, $\dim V = n$.

   (b) Let $A = \begin{bmatrix} 1 & 1 & k \\ 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Find all $k \in \mathbb{R}$ for which the columns of A are linearly dependent.

   (c) State the subspace axioms for a subset W of a vector space V and show that $N(A)$ is a subspace of $\mathbb{R}^n$ if $A \in M_{m\times n}(\mathbb{R})$.

   (d) Show that the set $W = \{(x_1, x_2, x_3, x_4)^t \mid x_2 = x_4 \text{ and } x_1 = x_3\}$ forms a subspace of $\mathbb{R}^4$. Find $\dim W$.

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3. (a) Let U and V be vector spaces and $T : U \to V$ be a linear transformation. Define the terms $\operatorname{Ker} T, \operatorname{Im} T, \operatorname{rank} T, \operatorname{nullity} T$ and state a connection between $\operatorname{rank} T$ and $\operatorname{nullity} T$.

   (b) Let $V = M_{2\times 2}(\mathbb{R})$ and let $T : V \to \mathbb{R}$ be the mapping defined by

$$T(A) = a_{11} + a_{22}, \text{ where } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

   (i) Prove that T is a linear transformation.

   (ii) Find a basis for $\operatorname{Ker} T$.

   (iii) Determine $\operatorname{rank} T$ and $\operatorname{nullity} T$.

4. (a) Define the terms *eigenvalue, eigenvector, eigenspace, geometric multiplicity, algebraic multiplicity.*

   (b) Let $A = \begin{bmatrix} 9 & 4 & -3 \\ -2 & 0 & 6 \\ -1 & -4 & 11 \end{bmatrix} \in M_{3\times 3}(\mathbb{R})$.

   (i) Verify that $ch_A(x) = (x-8)^2(x-4)$.

   (ii) Prove that A is diagonisable over $\mathbb{R}$.

   (iii) Find a non–singular matrix $P \in M_{3\times 3}(\mathbb{R})$ such that $P^{-1}AP = \operatorname{diag}(4, 8, 8)$.

5. (a) Let $A = \begin{bmatrix} 0 & 4 & -3 \\ -1 & -5 & 3 \\ -1 & -4 & 2 \end{bmatrix} \in M_{3\times 3}(\mathbb{R})$.

   (i) Given that $ch_A(x) = (x+1)^3$, prove that A is not diagonizable over $\mathbb{R}$.

   (ii) Find a 3×3 non–singular matrix P and a block upper triangular matrix B over $\mathbb{R}$, such that $P^{-1}AP = B$. (Note: $(A + I_3)^2 = 0$.)

   (b) Also find the Jordan canonical form J_A, together with a non–singular matrix P such that $P^{-1}AP = J_A$.

6. A mapping $T : P_2[\mathbb{R}] \to \mathbb{R}^3$ is defined by

$$T(a + bx + cx^2) = \begin{bmatrix} a+b+c \\ a \\ a-b+c \end{bmatrix},$$

where $P_2[\mathbb{R}]$ is the vector space of polynomials $a + bx + cx^2$.

   (a) Prove that T is a linear transformation.

   (b) Prove that T is an isomorphism between $P_2[\mathbb{R}]$ and $\mathbb{R}^3$.

   (c) If $S : \mathbb{R}^3 \to P_2[\mathbb{R}]$ is the inverse linear transformation, find a formula for $S(X)$, when $X = [a, b, c]^t$.

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7. (a) If u and v are non–zero orthogonal vectors in a real inner product space V, prove that u and v are linearly independent.

   (b) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors $u_1 = (1, 1, 0, 1)^t, (3, 1, 1, -1)^t, (0, 1, -1, 1)^t$.

8. (Compulsory question for MP274 students, but may be attempted by MP204 students)

   (a) Let A, P, Q be matrices of sizes, $m \times n, m \times m$ and $n \times n$, where P and Q are non–singular. Let $B = PAQ$.

   (i) If $Y = Q^{-1}X$ and $X \in N(A)$, show that $Y \in N(B)$.

   (ii) Let $T : N(A) \to N(B)$ be the linear mapping defined by $T(X) = Q^{-1}X$. Prove that T is an isomorphism between $N(A)$ and $N(B)$ and deduce that $\operatorname{rank} A = \operatorname{rank} B$.

   (b) If λ_1, λ_2 are distinct eigenvalues of a matrix A and X_1, X_2 are corresponding eigenvectors, prove that X_1 and X_2 are linearly independent.