Section 8.8

1. The given line has equations

$$\begin{aligned} x &= 3 + t(13 - 3) = 3 + 10t, \\ y &= -2 + t(3 + 2) = -2 + 5t, \\ z &= 7 + t(-8 - 7) = 7 - 15t. \end{aligned}$$

The line meets the plane $y = 0$ in the point $(x, 0, z)$, where $0 = -2 + 5t$, or $t = 2/5$. The corresponding values for x and z are 7 and 1, respectively.

2. $\mathbf{E} = \frac{1}{2}(\mathbf{B} + \mathbf{C})$, $\mathbf{F} = (1 - t)\mathbf{A} + t\mathbf{E}$, where

$$t = \frac{AF}{AE} = \frac{AF}{AF + FE} = \frac{AF/FE}{(AF/FE) + 1} = \frac{2}{3}.$$

Hence

$$\begin{aligned} \mathbf{F} &= \frac{1}{3}\mathbf{A} + \frac{2}{3}\left(\frac{1}{2}(\mathbf{B} + \mathbf{C})\right) \\ &= \frac{1}{3}\mathbf{A} + \frac{1}{3}(\mathbf{B} + \mathbf{C}) \\ &= \frac{1}{3}(\mathbf{A} + \mathbf{B} + \mathbf{C}). \end{aligned}$$

3. Let $A = (2, 1, 4)$, $B = (1, -1, 2)$, $C = (3, 3, 6)$. Then we prove $\overrightarrow{AC} = t\,\overrightarrow{AB}$ for some real t. We have

$$\overrightarrow{AC} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \overrightarrow{AB} = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}.$$

Hence $\overrightarrow{AC} = (-1)\,\overrightarrow{AB}$ and consequently C is on the line AB. In fact A is between C and B, with $AC = AB$.

4. The points P on the line AB which satisfy $AP = \frac{2}{5}PB$ are given by $\mathbf{P} = \mathbf{A} + t\,\overrightarrow{AB}$, where $|t/(1 - t)| = 2/5$. Hence $t/(1 - t) = \pm 2/5$.

The equation $t/(1 - t) = 2/5$ gives $t = 2/7$ and hence

$$\mathbf{P} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + \frac{2}{7}\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 16/7 \\ 29/7 \\ 3/7 \end{bmatrix}.$$

Hence $P = (16/7,\ 29/7,\ 3/7)$.

The equation $t/(1-t) = -2/5$ gives $t = -2/3$ and hence

$$\mathbf{P} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1/3 \\ -13/3 \end{bmatrix}.$$

Hence $P = (4/3,\ 1/3,\ -13/3)$.

5. An equation for $\mathcal{M}$ is $\mathbf{P} = \mathbf{A} + t\,\overrightarrow{BC}$, which reduces to

$$\begin{aligned} x &= 1 + 6t \\ y &= 2 - 3t \\ z &= 3 + 7t. \end{aligned}$$

An equation for $\mathcal{N}$ is $\mathbf{Q} = \mathbf{E} + s\,\overrightarrow{EF}$, which reduces to

$$\begin{aligned} x &= 1 + 9s \\ y &= -1 \\ z &= 8 + 3s. \end{aligned}$$

To find if and where $\mathcal{M}$ and $\mathcal{N}$ intersect, we set $P = Q$ and attempt to solve for s and t. We find the unique solution $t = 1,\ s = 2/3$, proving that the lines meet in the point

$$(x,\ y,\ z) = (1+6,\ 2-3,\ 3+7) = (7,\ -1,\ 10).$$

6. Let $A = (-3,\ 5,\ 6),\ B = (-2,\ 7,\ 9),\ C = (2,\ 1,\ 7)$. Then

(i)

$$\cos\angle ABC = (\overrightarrow{BA} \cdot \overrightarrow{BC})/(BA \cdot BC),$$

where $\overrightarrow{BA} = [-1,\ -2,\ -3]^t$ and $\overrightarrow{BC} = [4,\ -6,\ -2]^t$. Hence

$$\cos\angle ABC = \frac{-4+12+6}{\sqrt{14}\sqrt{56}} = \frac{14}{\sqrt{14}\sqrt{56}} = \frac{1}{2}.$$

Hence $\angle ABC = \pi/3$ radians or $60°$.

(ii)

$$\cos\angle BAC = (\overrightarrow{AB}\cdot\overrightarrow{AC})/(AB\cdot AC),$$

where $\overrightarrow{AB}= [1,\ 2,\ 3]^t$ and $\overrightarrow{AC}= [5,\ -4,\ 1]^t$. Hence

$$\cos\angle BAC = \frac{5-8+3}{\sqrt{14}\sqrt{42}} = 0.$$

Hence $\angle ABC = \pi/2$ radians or $90°$.

(iii)

$$\cos\angle ACB = (\overrightarrow{CA}\cdot\overrightarrow{CB})/(CA\cdot CB),$$

where $\overrightarrow{CA}= [-5,\ 4,\ -1]^t$ and $\overrightarrow{CB}= [-4,\ 6,\ 2]^t$. Hence

$$\cos\angle ACB = \frac{20+24-2}{\sqrt{42}\sqrt{56}} = \frac{42}{\sqrt{42}\sqrt{56}} = \frac{\sqrt{42}}{\sqrt{56}} = \frac{\sqrt{3}}{2}.$$

Hence $\angle ACB = \pi/6$ radians or $30°$.

7. By Theorem 8.5.2, the closest point P on the line AB to the origin O is given by $\mathbf{P} = \mathbf{A} + t\,\overrightarrow{AB}$, where

$$t = \frac{\overrightarrow{AO}\cdot\overrightarrow{AB}}{AB^2} = \frac{-\mathbf{A}\cdot\overrightarrow{AB}}{AB^2}.$$

Now

$$\mathbf{A}\cdot\overrightarrow{AB}= \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = -2.$$

Hence $t = 2/11$ and

$$\mathbf{P} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + \frac{2}{11}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -16/11 \\ 13/11 \\ 35/11 \end{bmatrix}$$

and $P = (-16/11,\ 13/11, 35/11)$.

Consequently the shortest distance OP is given by

$$\sqrt{\left(\frac{-16}{11}\right)^2 + \left(\frac{13}{11}\right)^2 + \left(\frac{35}{11}\right)^2} = \frac{\sqrt{1650}}{11} = \frac{\sqrt{15\times 11\times 10}}{11} = \frac{\sqrt{150}}{\sqrt{11}}.$$

Alternatively, we can calculate the distance OP^2, where P is an arbitrary point on the line AB and then minimize OP^2:

$$\mathbf{P} = \mathbf{A} + t\,\overrightarrow{AB} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+3t \\ 1+t \\ 3+t \end{bmatrix}.$$

Hence

$$\begin{aligned} OP^2 &= (-2+3t)^2 + (1+t)^2 + (3+t)^2 \\ &= 11t^2 - 4t + 14 \\ &= 11\left(t^2 - \frac{4}{11}t + \frac{14}{11}\right) \\ &= 11\left(\left\{t - \frac{2}{11}\right\}^2 + \frac{14}{11} - \frac{4}{121}\right) \\ &= 11\left(\left\{t - \frac{2}{11}\right\}^2 + \frac{150}{121}\right). \end{aligned}$$

Consequently

$$OP^2 \geq 11 \times \frac{150}{121}$$

for all t; moreover

$$OP^2 = 11 \times \frac{150}{121}$$

when $t = 2/11$.

8. We first find parametric equations for $\mathcal{N}$ by solving the equations

$$\begin{aligned} x + y - 2z &= 1 \\ x + 3y - z &= 4. \end{aligned}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 1 & 3 & -1 & 4 \end{bmatrix},$$

which reduces to

$$\begin{bmatrix} 1 & 0 & -5/2 & -1/2 \\ 0 & 1 & 1/2 & 3/2 \end{bmatrix}.$$

Hence $x = -\frac{1}{2} + \frac{5}{2}z$, $y = \frac{3}{2} - \frac{z}{2}$, with z arbitrary. Taking $z = 0$ gives a point $A = (-\frac{1}{2},\, \frac{3}{2},\, 0)$, while $z = 1$ gives a point $B = (2,\, 1,\, 1)$.

Hence if $C = (1, 0, 1)$, then the closest point on $\mathcal{N}$ to C is given by $\mathbf{P} = \mathbf{A} + t\,\overrightarrow{AB}$, where $t = (\overrightarrow{AC} \cdot \overrightarrow{AB})/AB^2$.

Now

$$\overrightarrow{AC} = \begin{bmatrix} 3/2 \\ -3/2 \\ 1 \end{bmatrix} \quad \text{and} \quad \overrightarrow{AB} = \begin{bmatrix} 5/2 \\ -1/2 \\ 1 \end{bmatrix},$$

so

$$t = \frac{\frac{3}{2} \times \frac{5}{2} + \frac{-3}{2} \times \frac{-1}{2} + 1 \times 1}{\left(\frac{5}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + 1^2} = \frac{11}{15}.$$

Hence

$$\mathbf{P} = \begin{bmatrix} -1/2 \\ 3/2 \\ 0 \end{bmatrix} + \frac{11}{15} \begin{bmatrix} 5/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 17/15 \\ 11/15 \end{bmatrix},$$

so $P = (4/3,\ 17/15,\ 11/15)$.

Also the shortest distance PC is given by

$$PC = \sqrt{\left(1 - \frac{4}{3}\right)^2 + \left(0 - \frac{17}{15}\right)^2 + \left(1 - \frac{11}{15}\right)^2} = \frac{\sqrt{330}}{15}.$$

9. The intersection of the planes $x + y - 2z = 4$ and $3x - 2y + z = 1$ is the line given by the equations

$$x = \frac{9}{5} + \frac{3}{5}z,\ y = \frac{11}{5} + \frac{7}{5}z,$$

where z is arbitrary. Hence the line $\mathcal{L}$ has a direction vector $[3/5,\ 7/5,\ 1]^t$ or the simpler $[3,\ 7,\ 5]^t$. Then any plane of the form $3x + 7y + 5z = d$ will be perpendicualr to $\mathcal{L}$. The required plane has to pass through the point $(6,\ 0,\ 2)$, so this determines d:

$$3 \times 6 + 7 \times 0 + 5 \times 2 = d = 28.$$

10. The length of the projection of the segment AB onto the line CD is given by the formula

$$\frac{|\ \overrightarrow{CD} \cdot \overrightarrow{AB}\ |}{CD}.$$

Here $\overrightarrow{CD} = [-8,\ 4,\ -1]^t$ and $\overrightarrow{AB} = [4,\ -4,\ 3]^t$, so

$$\begin{aligned} \frac{|\ \overrightarrow{CD} \cdot \overrightarrow{AB}\ |}{CD} &= \frac{|(-8) \times 4 + 4 \times (-4) + (-1) \times 3|}{\sqrt{(-8)^2 + 4^2 + (-1)^2}} \\ &= \frac{|-51|}{\sqrt{81}} = \frac{51}{9} = \frac{17}{3}. \end{aligned}$$

11. A direction vector for $\mathcal{L}$ is given by $\overrightarrow{BC}= [-5,\ -2,\ 3]^t$. Hence the plane through A perpendicular to $\mathcal{L}$ is given by

$$-5x - 2y + 3z = (-5)\times 3 + (-2)\times(-1) + 3\times 2 = -7.$$

The position vector $\mathbf{P}$ of an arbitrary point P on $\mathcal{L}$ is given by $\mathbf{P} = \mathbf{B} + t\ \overrightarrow{BC}$, or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} -5 \\ -2 \\ 3 \end{bmatrix},$$

or equivalently $x = 2 - 5t$, $y = 1 - 2t$, $z = 4 + 3t$.

To find the intersection of line $\mathcal{L}$ and the given plane, we substitute the expressions for x, y, z found in terms of t into the plane equation and solve the resulting linear equation for t:

$$-5(2 - 5t) - 2(1 - 2t) + 3(4 + 3t) = -7,$$

which gives $t = -7/38$. Hence $P = \left(\frac{111}{38}, \frac{52}{38}, \frac{131}{38}\right)$ and

$$\begin{aligned} AP &= \sqrt{\left(3 - \frac{111}{38}\right)^2 + \left(-1 - \frac{52}{38}\right)^2 + \left(2 - \frac{131}{38}\right)^2} \\ &= \frac{\sqrt{11134}}{38} = \frac{\sqrt{293\times 38}}{38} = \frac{\sqrt{293}}{\sqrt{38}}. \end{aligned}$$

12. Let P be a point inside the triangle ABC. Then the line through P and parallel to AC will meet the segments AB and BC in D and E, respectively. Then

$$\begin{aligned} \mathbf{P} &= (1 - r)\mathbf{D} + r\mathbf{E}, \quad 0 < r < 1; \\ \mathbf{D} &= (1 - s)\mathbf{B} + s\mathbf{A}, \quad 0 < s < 1; \\ \mathbf{E} &= (1 - t)\mathbf{B} + t\mathbf{C}, \quad 0 < t < 1. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{P} &= (1 - r)\{(1 - s)\mathbf{B} + s\mathbf{A}\} + r\{(1 - t)\mathbf{B} + t\mathbf{C}\} \\ &= (1 - r)s\mathbf{A} + \{(1 - r)(1 - s) + r(1 - t)\}\mathbf{B} + rt\mathbf{C} \\ &= \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}, \end{aligned}$$

where

$$\alpha = (1-r)s, \quad \beta = (1-r)(1-s) + r(1-t), \quad \gamma = rt.$$

Then $0 < \alpha < 1,\ 0 < \gamma < 1,\ 0 < \beta < (1-r) + r = 1$. Also

$$\alpha + \beta + \gamma = (1-r)s + (1-r)(1-s) + r(1-t) + rt = 1.$$

13. The line AB is given by $\mathbf{P} = \mathbf{A} + t[3,\, 4,\, 5]^t$, or

$$x = 6 + 3t, \quad y = -1 + 4t, \quad z = 11 + 5t.$$

Then B is found by substituting these expressions in the plane equation

$$3x + 4y + 5z = 10.$$

We find $t = -59/50$ and consequently

$$B = \left(6 - \frac{177}{50},\, -1 - \frac{236}{50},\, 11 - \frac{295}{50}\right) = \left(\frac{123}{50},\, \frac{-286}{50},\, \frac{255}{50}\right).$$

Then

$$\begin{aligned} AB &= ||\,\overrightarrow{AB}\,|| = ||t \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}|| \\ &= |t|\sqrt{3^2 + 4^2 + 5^2} = \frac{59}{50} \times \sqrt{50} = \frac{59}{\sqrt{50}}. \end{aligned}$$

14. Let $A = (-3,\, 0,\, 2),\ B = (6,\, 1,\, 4),\ C = (-5,\, 1,\, 0)$. Then the area of triangle ABC is $\frac{1}{2}||\,\overrightarrow{AB} \times \overrightarrow{AC}\,||$. Now

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 9 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 14 \\ 11 \end{bmatrix}.$$

Hence $||\,\overrightarrow{AB} \times \overrightarrow{AC}\,|| = \sqrt{333}$.

15. Let $A_1 = (2,\, 1,\, 4),\ A_2 = (1,\, -1,\, 2),\ A_3 = (4,\, -1,\, 1)$. Then the point $P = (x,\, y,\, z)$ lies on the plane $A_1A_2A_3$ if and only if

$$\overrightarrow{A_1P} \cdot (\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3}) = 0,$$

or

$$\begin{vmatrix} x-2 & y-1 & z-4 \\ -1 & -2 & -2 \\ 2 & -2 & -3 \end{vmatrix} = 2x - 7y + 6z - 21 = 0.$$

16. Non–parallel lines $\mathcal{L}$ and $\mathcal{M}$ in three dimensional space are given by equations

$$\mathbf{P} = \mathbf{A} + sX, \quad \mathbf{Q} = \mathbf{B} + tY.$$

(i) Suppose $\overrightarrow{PQ}$ is orthogonal to both X and Y. Now

$$\overrightarrow{PQ} = \mathbf{Q} - \mathbf{P} = (\mathbf{B} + tY) - (\mathbf{A} + sX) = \overrightarrow{AB} + tY - sX.$$

Hence

$$\begin{aligned} (\overrightarrow{AB} + tY + sX) \cdot X &= 0 \\ (\overrightarrow{AB} + tY + sX) \cdot Y &= 0. \end{aligned}$$

More explicitly

$$\begin{aligned} t(Y \cdot X) - s(X \cdot X) &= -\overrightarrow{AB} \cdot X \\ t(Y \cdot Y) - s(X \cdot Y) &= -\overrightarrow{AB} \cdot Y. \end{aligned}$$

However the coefficient determinant of this system of linear equations in t and s is equal to

$$\begin{aligned} \begin{vmatrix} Y \cdot X & -X \cdot X \\ Y \cdot Y & -X \cdot Y \end{vmatrix} &= -(X \cdot Y)^2 + (X \cdot X)(Y \cdot Y) \\ &= \|X \times Y\|^2 \neq 0, \end{aligned}$$

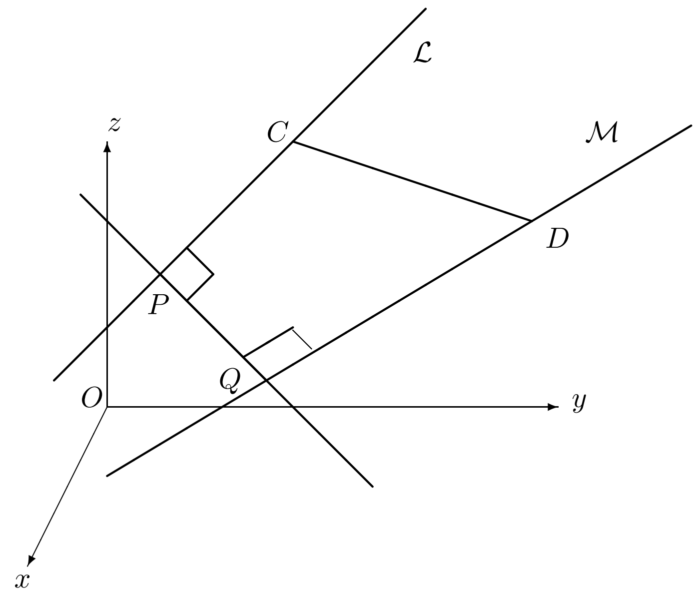
as $X \neq 0$, $Y \neq 0$ and X and Y are not proportional ($\mathcal{L}$ and $\mathcal{M}$ are not parallel).

(ii) P and Q can be viewed as the projections of C and D onto the line PQ, where C and D are arbitrary points on the lines $\mathcal{L}$ and $\mathcal{M}$, respectively. Hence by equation (8.14) of Theorem 8.5.3, we have

$$PQ \leq CD.$$

Finally we derive a useful formula for PQ. Again by Theorem 8.5.3

$$PQ = \frac{|\overrightarrow{AB} \cdot \overrightarrow{PQ}|}{PQ} = |\overrightarrow{AB} \cdot \hat{n}|,$$



where $\hat{n} = \frac{1}{PQ}\ \overrightarrow{PQ}$ is a unit vector which is orthogonal to X and Y. Hence

$$\hat{n} = t(X \times Y),$$

where $t = \pm 1/||X \times Y||$. Hence

$$PQ = \frac{|\ \overrightarrow{AB} \cdot (X \times Y)|}{||X \times Y||}.$$

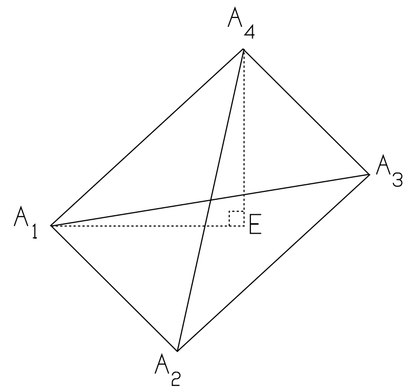
17. We use the formula of the previous question.
Line $\mathcal{L}$ has the equation $\mathbf{P} = \mathbf{A} + sX$, where

$$X = \overrightarrow{AC} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}.$$

Line $\mathcal{M}$ has the equation $\mathbf{Q} = \mathbf{B} + tY$, where

$$Y = \overrightarrow{BD} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence $X \times Y = [-6,\ 1,\ 5]^t$ and $||X \times Y|| = \sqrt{62}$.



Hence the shortest distance between lines AC and BD is equal to

$$\frac{|\overrightarrow{AB}\cdot(X\times Y)|}{||X\times Y||}=\frac{\left|\begin{bmatrix}0\\-2\\1\end{bmatrix}\cdot\begin{bmatrix}-6\\1\\5\end{bmatrix}\right|}{\sqrt{62}}=\frac{3}{\sqrt{62}}.$$

18. Let E be the foot of the perpendicular from A_4 to the plane $A_1A_2A_3$. Then

$$\mathrm{vol}\,A_1A_2A_3A_4=\frac{1}{3}(\,\mathrm{area}\,\Delta A_1A_2A_3)\cdot A_4E.$$

Now

$$\mathrm{area}\,\Delta A_1A_2A_3=\frac{1}{2}||\,\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}\,||.$$

Also A_4E is the length of the projection of A_1A_4 onto the line A_4E. (See figure above.)

Hence $A_4E=|\,\overrightarrow{A_1A_4}\cdot X|$, where X is a unit direction vector for the line A_4E. We can take

$$X=\frac{\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}}{||\,\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}\,||}.$$

Hence

$$\begin{aligned}\mathrm{vol}\,A_1A_2A_3A_4 &= \frac{1}{6}||\,\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}\,||\frac{|\,\overrightarrow{A_1A_4}\cdot(\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3})|}{||\,\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}\,||}\\ &= \frac{1}{6}|\,\overrightarrow{A_1A_4}\cdot(\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3})|\end{aligned}$$

$$= \frac{1}{6}|(\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3})\cdot \overrightarrow{A_1A_4}|.$$

19. We have $\overrightarrow{CB}= [1,\ 4,\ -1]^t$, $\overrightarrow{CD}= [-3,\ 3,\ 0]^t$, $\overrightarrow{AD}= [3,\ 0,\ 3]^t$. Hence

$$\overrightarrow{CB} \times \overrightarrow{CD}= 3\mathbf{i} + 3\mathbf{j} + 15\mathbf{k},$$

so the vector $\mathbf{i}+\mathbf{j}+5\mathbf{k}$ is perpendicular to the plane BCD.

Now the plane BCD has equation $x+y+5z = 9$, as $B = (2,\ 2,\ 1)$ is on the plane.

Also the line through A normal to plane BCD has equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = (1+t) \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}.$$

Hence $x = 1+t,\ y = 1+t,\ z = 5(1+t)$.

[We remark that this line meets plane BCD in a point E which is given by a value of t found by solving

$$(1+t) + (1+t) + 5(5+5t) = 9.$$

So $t = -2/3$ and $E = (1/3,\ 1/3,\ 5/3)$.]

The distance from A to plane BCD is

$$\frac{|1 \times 1 + 1 \times 1 + 5 \times 5 - 9|}{1^2 + 1^2 + 5^2} = \frac{18}{\sqrt{27}} = 2\sqrt{3}.$$

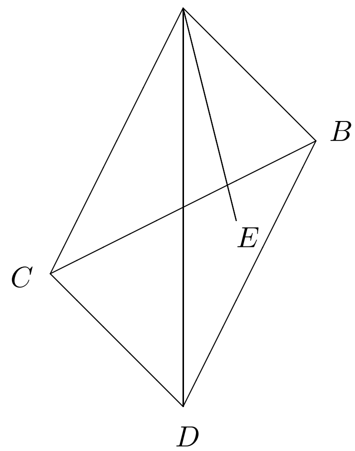
To find the distance between lines AD and BC, we first note that

(a) The equation of AD is

$$\mathbf{P} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+3t \\ 1 \\ 5+3t \end{bmatrix};$$

(b) The equation of BC is

$$\mathbf{Q} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+s \\ 2+4s \\ 1-s \end{bmatrix}.$$



Then $\overrightarrow{PQ} = [1 + s - 3t,\ 1 + 4s,\ -4 - s - 3t]^t$ and we find s and t by solving the equations $\overrightarrow{PQ} \cdot \overrightarrow{AD} = 0$ and $\overrightarrow{PQ} \cdot \overrightarrow{BC} = 0$, or

$$\begin{aligned} (1 + s - 3t)3 + (1 + 4s)0 + (-4 - s - 3t)3 &= 0 \\ (1 + s - 3t) + 4(1 + 4s) - (-4 - s - 3t) &= 0. \end{aligned}$$

Hence $t = -1/2 = s$.

Correspondingly, $P = (-1/2,\ 1,\ 7/2)$ and $Q = (3/2,\ 0,\ 3/2)$.

Thus we have found the closest points P and Q on the respective lines AD and BC. Finally the shortest distance between the lines is

$$PQ = || \overrightarrow{PQ} || = 3.$$