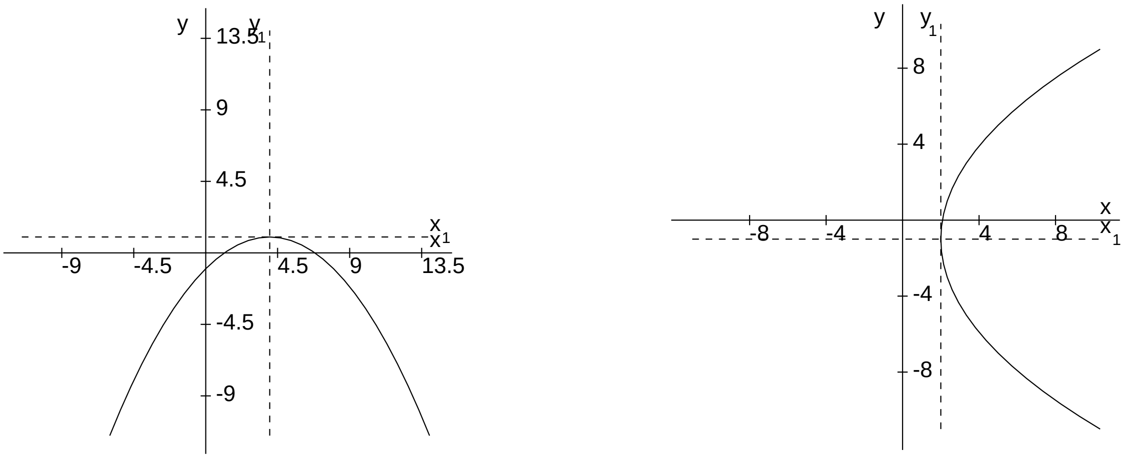

Figure 1: (a): $x^2 - 8x + 8y + 8 = 0$; (b): $y^2 - 12x + 2y + 25 = 0$

Section 7.3

1. (i) $x^2 - 8x + 8y + 8 = (x-4)^2 + 8(y-1)$. So the equation $x^2 - 8x + 8y + 8 = 0$ becomes

$$x_1^2 + 8y_1 = 0 \qquad (1)$$

if we make a translation of axes $x - 4 = x_1,\ y - 1 = y_1$.

However equation (1) can be written as a standard form

$$y_1 = -\frac{1}{8}x_1^2,$$

which represents a parabola with vertex at (4, 1). (See Figure 1(a).)

(ii) $y^2 - 12x + 2y + 25 = (y+1)^2 - 12(x-2)$. Hence $y^2 - 12x + 2y + 25 = 0$ becomes

$$y_1^2 - 12x_1 = 0 \qquad (2)$$

if we make a translation of axes $x - 2 = x_1,\ y + 1 = y_1$.

However equation (2) can be written as a standard form

$$y_1^2 = 12x_1,$$

which represents a parabola with vertex at (2, −1). (See Figure 1(b).)

2. $4xy - 3y^2 = X^tAX$, where $A = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$. The eigenvalues of A are the roots of $\lambda^2 + 3\lambda - 4 = 0$, namely $\lambda_1 = -4$ and $\lambda_2 = 1$.

The eigenvectors corresponding to an eigenvalue λ are the non–zero vectors $[x,\, y]^t$ satisfying

$$\begin{bmatrix} 0-\lambda & 2 \\ 2 & -3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\lambda_1 = -4$ gives equations

$$\begin{aligned} 4x + 2y &= 0 \\ 2x + y &= 0 \end{aligned}$$

which has the solution $y = -2x$. Hence

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix} = x\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

A corresponding unit eigenvector is $[1/\sqrt{5},\, -2/\sqrt{5}]^t$.

$\lambda_2 = 1$ gives equations

$$\begin{aligned} -x + 2y &= 0 \\ 2x - 4y &= 0 \end{aligned}$$

which has the solution $x = 2y$. Hence

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

A corresponding unit eigenvector is $[2/\sqrt{5},\, 1/\sqrt{5}]^t$.

Hence if

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix},$$

then P is an orthogonal matrix. Also as $\det P = 1$, P is a proper orthogonal matrix and the equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = P\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

represents a rotation to new x_1, y_1 axes whose positive directions are given by the respective columns of P. Also

$$P^tAP = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $X^tAX = -4x_1^2 + y_1^2$ and the original equation $4xy - 3y^2 = 8$ becomes $-4x_1^2 + y_1^2 = 8$, or the standard form

$$\frac{-x_1^2}{2} + \frac{y_1^2}{8} = 1,$$

which represents an hyperbola.

The asymptotes assist in drawing the curve. They are given by the equations

$$\frac{-x_1^2}{2} + \frac{y_1^2}{8} = 0, \quad \text{or} \quad y_1 = \pm 2x_1.$$

Now

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = P^t \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

so

$$x_1 = \frac{x - 2y}{\sqrt{5}}, \quad y_1 = \frac{2x + y}{\sqrt{5}}.$$

Hence the asymptotes are

$$\frac{2x + y}{\sqrt{5}} = \pm 2 \left(\frac{x - 2y}{\sqrt{5}} \right),$$

which reduces to $y = 0$ and $y = 4x/3$. (See Figure 2(a).)

3. $8x^2 - 4xy + 5y^2 = X^tAX$, where $A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$. The eigenvalues of A are the roots of $\lambda^2 - 13\lambda + 36 = 0$, namely $\lambda_1 = 4$ and $\lambda_2 = 9$. Corresponding unit eigenvectors turn out to be $[1/\sqrt{5}, 2/\sqrt{5}]^t$ and $[-2/\sqrt{5}, 1/\sqrt{5}]^t$. Hence if

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix},$$

then P is an orthogonal matrix. Also as $\det P = 1$, P is a proper orthogonal matrix and the equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

represents a rotation to new x_1, y_1 axes whose positive directions are given by the respective columns of P. Also

$$P^tAP = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.$$

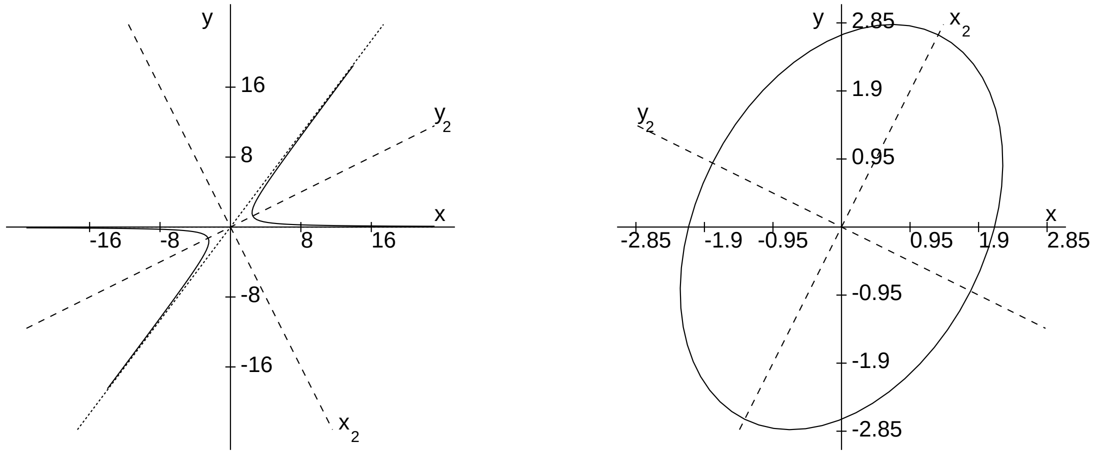

Figure 2: (a): $4xy - 3y^2 = 8$; (b): $8x^2 - 4xy + 5y^2 = 36$

Then $X^tAX = 4x_1^2 + 9y_1^2$ and the original equation $8x^2 - 4xy + 5y^2 = 36$ becomes $4x_1^2 + 9y_1^2 = 36$, or the standard form

$$\frac{x_1^2}{9} + \frac{y_1^2}{4} = 1,$$

which represents an ellipse as in Figure 2(b).

The axes of symmetry turn out to be $y = 2x$ and $x = -2y$.

4. We give the sketch only for parts (i), (iii) and (iv). We give the working for (ii) only. See Figures 3(a) and 4(a) and 4(b), respectively.

(ii) We have to investigate the equation

$$5x^2 - 4xy + 8y^2 + 4\sqrt{5}x - 16\sqrt{5}y + 4 = 0. \qquad (3)$$

Here $5x^2 - 4xy + 8y^2 = X^tAX$, where $A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$. The eigenvalues of A are the roots of $\lambda^2 - 13\lambda + 36 = 0$, namely $\lambda_1 = 9$ and $\lambda_2 = 4$. Corresponding unit eigenvectors turn out to be $[1/\sqrt{5}, -2/\sqrt{5}]^t$ and $[2/\sqrt{5}, 1/\sqrt{5}]^t$. Hence if

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix},$$

then P is an orthogonal matrix. Also as $\det P = 1$, P is a proper orthogonal matrix and the equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

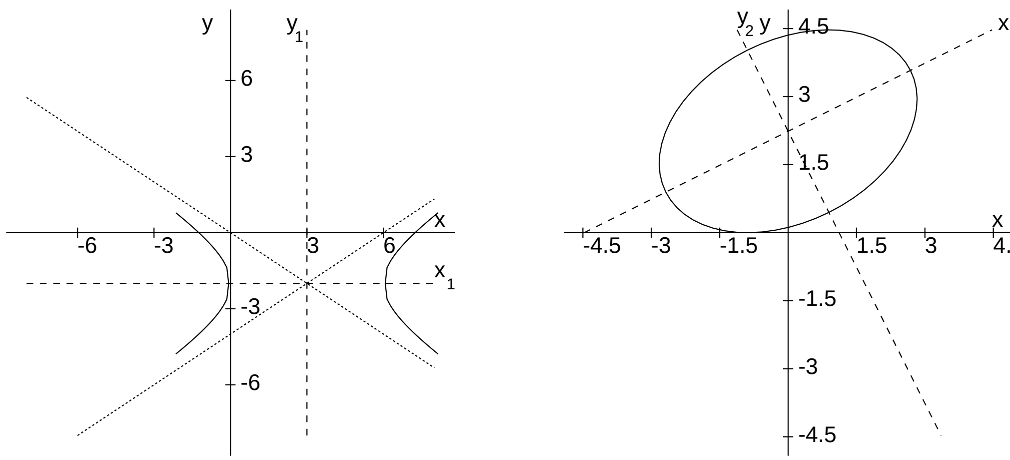

Figure 3: (a): $4x^2 - 9y^2 - 24x - 36y - 36 = 0$; (b): $5x^2 - 4xy + 8y^2 + \sqrt{5}x - 16\sqrt{5}y + 4 = 0$

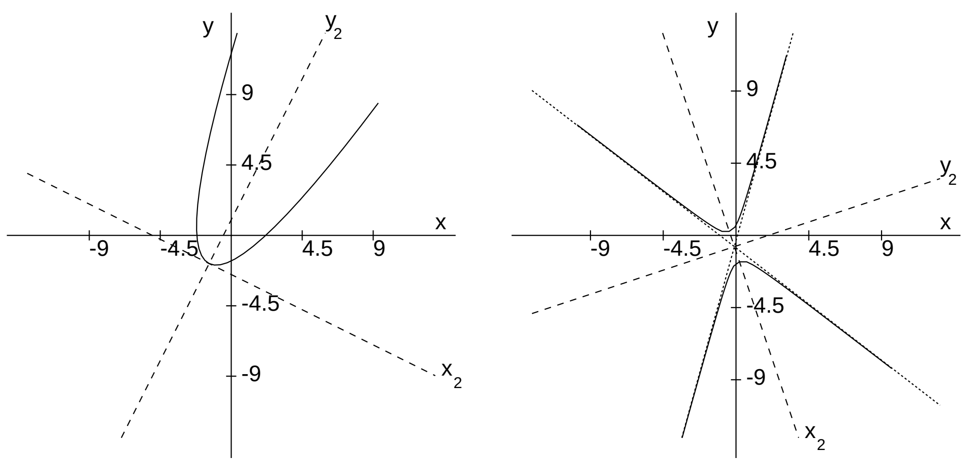

Figure 4: (a): $4x^2 + y^2 - 4xy - 10y - 19 = 0$; (b): $77x^2 + 78xy - 27y^2 + 70x - 30y + 29 = 0$

represents a rotation to new x_1, y_1 axes whose positive directions are given by the respective columns of P. Also

$$P^tAP = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}.$$

Moreover

$$5x^2 - 4xy + 8y^2 = 9x_1^2 + 4y_1^2.$$

To get the coefficients of x_1 and y_1 in the transformed form of equation (3), we have to use the rotation equations

$$x = \frac{1}{\sqrt{5}}(x_1 + 2y_1), \quad y = \frac{1}{\sqrt{5}}(-2x_1 + y_1).$$

Then equation (3) transforms to

$$9x_1^2 + 4y_1^2 + 36x_1 - 8y_1 + 4 = 0,$$

or, on completing the square,

$$9(x_1 + 2)^2 + 4(y_1 - 1)^2 = 36,$$

or in standard form

$$\frac{x_2^2}{4} + \frac{y_2^2}{9} = 1,$$

where $x_2 = x_1 + 2$ and $y_2 = y_1 - 1$. Thus we have an ellipse, centre $(x_2, y_2) = (0, 0)$, or $(x_1, y_1) = (-2, 1)$, or $(x, y) = (0, \sqrt{5})$.

The axes of symmetry are given by $x_2 = 0$ and $y_2 = 0$, or $x_1 + 2 = 0$ and $y_1 - 1 = 0$, or

$$\frac{1}{\sqrt{5}}(x - 2y) + 2 = 0 \quad \text{and} \quad \frac{1}{\sqrt{5}}(2x + y) - 1 = 0,$$

which reduce to $x - 2y + 2\sqrt{5} = 0$ and $2x + y - \sqrt{5} = 0$. See Figure 3(b).

5. (i) Consider the equation

$$2x^2 + y^2 + 3xy - 5x - 4y + 3 = 0. \qquad (4)$$

$$\Delta = \begin{vmatrix} 2 & 3/2 & -5/2 \\ 3/2 & 1 & -2 \\ -5/2 & -2 & 3 \end{vmatrix} = 8\begin{vmatrix} 4 & 3 & -5 \\ 3 & 2 & -4 \\ -5 & -4 & 6 \end{vmatrix} = 8\begin{vmatrix} 1 & 1 & -1 \\ 3 & 2 & -4 \\ -2 & -2 & 2 \end{vmatrix} = 0.$$

Let $x = x_1 + \alpha$, $y = y_1 + \beta$ and substitute in equation (4) to get

$$2(x_1+\alpha)^2+(y_1+\beta)^2+3(x_1+\alpha)(y_1+\beta)-5(x_1+\alpha)-4(y_1+\beta)+3=0 \quad (5).$$

Then equating the coefficients of x_1 and y_1 to 0 gives

$$\begin{aligned} 4\alpha + 3\beta - 5 &= 0 \\ 3\alpha + 2\beta - 4 &= 0, \end{aligned}$$

which has the unique solution $\alpha = 2$, $\beta = -1$. Then equation (5) simplifies to

$$2x_1^2 + y_1^2 + 3x_1y_1 = 0 = (2x_1 + y_1)(x_1 + y_1).$$

So relative to the x_1, y_1 coordinates, equation (4) describes two lines: $2x_1 + y_1 = 0$ and $x_1 + y_1 = 0$. In terms of the original x, y coordinates, these lines become $2(x-2)+(y+1) = 0$ and $(x-2)+(y+1) = 0$, i.e. $2x + y - 3 = 0$ and $x + y - 1 = 0$, which intersect in the point

$$(x, y) = (\alpha, \beta) = (2, -1).$$

(ii) Consider the equation

$$9x^2 + y^2 - 6xy + 6x - 2y + 1 = 0. \quad (6)$$

Here

$$\Delta = \begin{vmatrix} 9 & -3 & 3 \\ 3 & 1 & -1 \\ 3 & -1 & 1 \end{vmatrix} = 0,$$

as column 3 = − column 2.

Let $x = x_1 + \alpha$, $y = y_1 + \beta$ and substitute in equation (6) to get

$$9(x_1+\alpha)^2 + (y_1+\beta)^2 - 6(x_1+\alpha)(y_1+\beta) + 6(x_1+\alpha) - 2(y_1+\beta) + 1 = 0.$$

Then equating the coefficients of x_1 and y_1 to 0 gives

$$\begin{aligned} 18\alpha - 6\beta + 6 &= 0 \\ -6\alpha + 2\beta - 2 &= 0, \end{aligned}$$

or equivalently $-3\alpha + \beta - 1 = 0$. Take $\alpha = 0$ and $\beta = 1$. Then equation (6) simplifies to

$$9x_1^2 + y_1^2 - 6x_1y_1 = 0 = (3x_1 - y_1)^2. \quad (7)$$

In terms of x, y coordinates, equation (7) becomes

$$(3x - (y-1))^2 = 0, \text{ or } 3x - y + 1 = 0.$$

(iii) Consider the equation

$$x^2 + 4xy + 4y^2 - x - 2y - 2 = 0. \tag{8}$$

Arguing as in the previous examples, we find that any translation

$$x = x_1 + \alpha, \quad y = y_1 + \beta$$

where $2\alpha + 4\beta - 1 = 0$ has the property that the coefficients of x_1 and y_1 will be zero in the transformed version of equation (8). Take $\beta = 0$ and $\alpha = 1/2$. Then (8) reduces to

$$x_1^2 + 4x_1y_1 + 4y_1^2 - \frac{9}{4} = 0,$$

or $(x_1 + 2y_1)^2 = 3/2$. Hence $x_1 + 2y_1 = \pm 3/2$, with corresponding equations

$$x + 2y = 2 \quad \text{and} \quad x + 2y = -1.$$