Section 6.3

1. Let $A = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$. Then A has characteristic equation $\lambda^2 - 4\lambda + 3 = 0$ or $(\lambda - 3)(\lambda - 1) = 0$. Hence the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 1$.

$\lambda_1 = 3$. The corresponding eigenvectors satisfy $(A - \lambda_1 I_2)X = 0$, or

$$\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or equivalently $x - 3y = 0$. Hence

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3y \\ y \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and we take $X_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Similarly for $\lambda_2 = 1$ we find the eigenvector $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Hence if $P = [X_1|X_2] = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$, then P is non–singular and

$$P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$A = P \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} P^{-1}$$

and consequently

$$\begin{aligned} A^n &= P \begin{bmatrix} 3^n & 0 \\ 0 & 1^n \end{bmatrix} P^{-1} \\ &= \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & 1^n \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 3^{n+1} & 1 \\ 3^n & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 3^{n+1} - 1 & -3^{n+1} + 3 \\ 3^n - 1 & -3^n + 3 \end{bmatrix} \\ &= \frac{3^n - 1}{2} A + \frac{3 - 3^n}{2} I_2. \end{aligned}$$

2. Let $A = \begin{bmatrix} 3/5 & 4/5 \\ 2/5 & 1/5 \end{bmatrix}$. Then we find that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1/5$, with corresponding eigenvectors

$$X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then if $P = [X_1|X_2]$, P is non–singular and

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1/5 \end{bmatrix} \quad \text{and} \quad A = P\begin{bmatrix} 1 & 0 \\ 0 & -1/5 \end{bmatrix}P^{-1}.$$

Hence

$$\begin{aligned} A^n &= P\begin{bmatrix} 1 & 0 \\ 0 & (-1/5)^n \end{bmatrix}P^{-1} \\ &\to P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1} \\ &= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}. \end{aligned}$$

3. The given system of differential equations is equivalent to $\dot{X} = AX$, where

$$A = \begin{bmatrix} 3 & -2 \\ 5 & -4 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The matrix $P = \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix}$ is a non-singular matrix of eigenvectors corresponding to eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 1$. Then

$$P^{-1}AP = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}.$$

The substitution $X = PY$, where $Y = [x_1, y_1]^t$, gives

$$\dot{Y} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}Y,$$

or equivalently $\dot{x_1} = -2x_1$ and $\dot{y_1} = y_1$.

Hence $x_1 = x_1(0)e^{-2t}$ and $y_1 = y_1(0)e^t$. To determine $x_1(0)$ and $y_1(0)$, we note that

$$\begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix} = P^{-1}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 13 \\ 22 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Hence $x_1 = 3e^{-2t}$ and $y_1 = 7e^t$. Consequently

$$x = 2x_1 + y_1 = 6e^{-2t} + 7e^t \quad \text{and} \quad y = 5x_1 + y_1 = 15e^{-2t} + 7e^t.$$

4. Introducing the vector $X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, the system of recurrence relations

$$\begin{aligned} x_{n+1} &= 3x_n - y_n \\ y_{n+1} &= -x_n + 3y_n, \end{aligned}$$

becomes $X_{n+1} = AX_n$, where $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$. Hence $X_n = A^n X_0$, where $X_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

To find A^n we can use the eigenvalue method. We get

$$A^n = \frac{1}{2}\begin{bmatrix} 2^n + 4^n & 2^n - 4^n \\ 2^n - 4^n & 2^n + 4^n \end{bmatrix}.$$

Hence

$$\begin{aligned} X_n &= \frac{1}{2}\begin{bmatrix} 2^n + 4^n & 2^n - 4^n \\ 2^n - 4^n & 2^n + 4^n \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 2^n + 4^n + 2(2^n - 4^n) \\ 2^n - 4^n + 2(2^n + 4^n) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 3 \times 2^n - 4^n \\ 3 \times 2^n + 4^n \end{bmatrix} = \begin{bmatrix} (3 \times 2^n - 4^n)/2 \\ (3 \times 2^n + 4^n)/2 \end{bmatrix}. \end{aligned}$$

Hence $x_n = \frac{1}{2}(3 \times 2^n - 4^n)$ and $y_n = \frac{1}{2}(3 \times 2^n + 4^n)$.

5. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a real or complex matrix with distinct eigenvalues λ_1, λ_2 and corresponding eigenvectors X_1, X_2. Also let $P = [X_1|X_2]$.

(a) The system of recurrence relations

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned}$$

has the solution

$$\begin{aligned}
\begin{bmatrix} x_n \\ y_n \end{bmatrix} &= A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \left(P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1} \right)^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&= P \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&= [X_1|X_2] \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\
&= [X_1|X_2] \begin{bmatrix} \lambda_1^n \alpha \\ \lambda_2^n \beta \end{bmatrix} = \lambda_1^n \alpha X_1 + \lambda_2^n \beta X_2,
\end{aligned}$$

where

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = P^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

(b) In matrix form, the system is $\dot{X} = AX$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$. We substitute $X = PY$, where $Y = [x_1, \, y_1]^t$. Then

$$\dot{X} = P\dot{Y} = AX = A(PY),$$

so

$$\dot{Y} = (P^{-1}AP)Y = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Hence $\dot{x_1} = \lambda_1 x_1$ and $\dot{y_1} = \lambda_2 y_1$. Then

$$x_1 = x_1(0)e^{\lambda_1 t} \quad \text{and} \quad y_1 = y_1(0)e^{\lambda_2 t}.$$

But

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = P \begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix},$$

so

$$\begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix} = P^{-1} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Consequently $x_1(0) = \alpha$ and $y_1(0) = \beta$ and

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= P \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = [X_1|X_2] \begin{bmatrix} \alpha e^{\lambda_1 t} \\ \beta e^{\lambda_2 t} \end{bmatrix} \\
&= \alpha e^{\lambda_1 t} X_1 + \beta e^{\lambda_2 t} X_2.
\end{aligned}$$

6. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a real matrix with non–real eigenvalues $\lambda = a + ib$ and $\overline{\lambda} = a - ib$, with corresponding eigenvectors $X = U + iV$ and $\overline{X} = U - iV$, where U and V are real vectors. Also let P be the real matrix defined by $P = [U|V]$. Finally let $a + ib = re^{i\theta}$, where $r > 0$ and θ is real.

(a) As X is an eigenvector corresponding to the eigenvalue λ, we have $AX = \lambda X$ and hence

$$\begin{aligned} A(U + iV) &= (a + ib)(U + iV) \\ AU + iAV &= aU - bV + i(bU + aV). \end{aligned}$$

Equating real and imaginary parts then gives

$$\begin{aligned} AU &= aU - bV \\ AV &= bU + aV. \end{aligned}$$

(b)

$$AP = A[U|V] = [AU|AV] = [aU - bV|bU + aV] = [U|V] \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = P \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Hence, as P can be shown to be non–singular,

$$P^{-1}AP = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

(The fact that P is non–singular is easily proved by showing the columns of P are linearly independent: Assume $xU + yV = 0$, where x and y are real. Then we find

$$(x + iy)(U - iV) + (x - iy)(U + iV) = 0.$$

Consequently $x + iy = 0$ as $U - iV$ and $U + iV$ are eigenvectors corresponding to distinct eigenvalues $a - ib$ and $a + ib$ and are hence linearly independent. Hence $x = 0$ and $y = 0$.)

(c) The system of recurrence relations

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned}$$

has solution

$$\begin{aligned}
\begin{bmatrix} x_n \\ y_n \end{bmatrix} &= A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&= P \begin{bmatrix} a & b \\ -b & a \end{bmatrix}^n P^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&= P \begin{bmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}^n \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\
&= Pr^n \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^n \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\
&= r^n [U|V] \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\
&= r^n [U|V] \begin{bmatrix} \alpha\cos n\theta + \beta\sin n\theta \\ -\alpha\sin n\theta + \beta\cos n\theta \end{bmatrix} \\
&= r^n \left\{(\alpha\cos n\theta + \beta\sin n\theta)U + (-\alpha\sin n\theta + \beta\cos n\theta)V\right\} \\
&= r^n \left\{(\cos n\theta)(\alpha U + \beta V) + (\sin n\theta)(\beta U - \alpha V)\right\}.
\end{aligned}$$

(d) The system of differential equations

$$\begin{aligned}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{aligned}$$

is attacked using the substitution $X = PY$, where $Y = [x_1,\, y_1]^t$. Then

$$\dot{Y} = (P^{-1}AP)Y,$$

so

$$\begin{bmatrix} \dot{x_1} \\ \dot{y_1} \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Equating components gives

$$\begin{aligned}
\dot{x_1} &= ax_1 + by_1 \\
\dot{y_1} &= -bx_1 + ay_1.
\end{aligned}$$

Now let $z = x_1 + iy_1$. Then

$$\begin{aligned}
\dot{z} = \dot{x_1} + i\dot{y_1} &= (ax_1 + by_1) + i(-bx_1 + ay_1) \\
&= (a - ib)(x_1 + iy_1) = (a - ib)z.
\end{aligned}$$

Hence

$$\begin{aligned} z &= z(0)e^{(a-ib)t} \\ x_1 + iy_1 &= (x_1(0) + iy_1(0))e^{at}(\cos bt - i\sin bt). \end{aligned}$$

Equating real and imaginary parts gives

$$\begin{aligned} x_1 &= e^{at}\left\{x_1(0)\cos bt + y_1(0)\sin bt\right\} \\ y_1 &= e^{at}\left\{y_1(0)\cos bt - x_1(0)\sin bt\right\}. \end{aligned}$$

Now if we define α and β by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = P^{-1}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix},$$

we see that $\alpha = x_1(0)$ and $\beta = y_1(0)$. Then

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= P\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\ &= [U|V]\begin{bmatrix} e^{at}(\alpha\cos bt + \beta\sin bt) \\ e^{at}(\beta\cos bt - \alpha\sin bt) \end{bmatrix} \\ &= e^{at}\{(\alpha\cos bt + \beta\sin bt)U + (\beta\cos bt - \alpha\sin bt)V\} \\ &= e^{at}\{\cos bt(\alpha U + \beta V) + \sin bt(\beta U - \alpha V)\}. \end{aligned}$$

7. (The case of repeated eigenvalues.) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and suppose that the characteristic polynomial of A, $\lambda^2 - (a+d)\lambda + (ad - bc)$, has a repeated root α. Also assume that $A \neq \alpha I_2$.

(i)

$$\begin{aligned} \lambda^2 - (a+d)\lambda + (ad-bc) &= (\lambda - \alpha)^2 \\ &= \lambda^2 - 2\alpha\lambda + \alpha^2. \end{aligned}$$

Hence $a + d = 2\alpha$ and $ad - bc = \alpha^2$ and

$$\begin{aligned} (a+d)^2 &= 4(ad - bc), \\ a^2 + 2ad + d^2 &= 4ad - 4bc, \\ a^2 - 2ad + d^2 + 4bc &= 0, \\ (a-d)^2 + 4bc &= 0. \end{aligned}$$

(ii) Let $B - A - \alpha I_2$. Then

$$\begin{aligned} B^2 = (A - \alpha I_2)^2 &= A^2 - 2\alpha A + \alpha^2 I_2 \\ &= A^2 - (a+d)A + (ad - bc)I_2, \end{aligned}$$

But by problem 3, chapter 2.4, $A^2 - (a+d)A + (ad-bc)I_2 = 0$, so $B^2 = 0$.

(iii) Now suppose that $B \neq 0$. Then $BE_1 \neq 0$ or $BE_2 \neq 0$, as BE_i is the i–th column of B. Hence $BX_2 \neq 0$, where $X_2 = E_1$ or $X_2 = E_2$.

(iv) Let $X_1 = BX_2$ and $P = [X_1|X_2]$. We prove P is non–singular by demonstrating that X_1 and X_2 are linearly independent.

Assume $xX_1 + yX_2 = 0$. Then

$$\begin{aligned} xBX_2 + yX_2 &= 0 \\ B(xBX_2 + yX_2) &= B0 = 0 \\ xB^2X_2 + yBX_2 &= 0 \\ x0X_2 + yBX_2 &= 0 \\ yBX_2 &= 0. \end{aligned}$$

Hence $y = 0$ as $BX_2 \neq 0$. Hence $xBX_2 = 0$ and so $x = 0$.

Finally, $BX_1 = B(BX_2) = B^2X_2 = 0$, so $(A - \alpha I_2)X_1 = 0$ and

$$AX_1 = \alpha X_1. \qquad (2)$$

Also

$$X_1 = BX_2 = (A - \alpha I_2)X_2 = AX_2 - \alpha X_2.$$

Hence

$$AX_2 = X_1 + \alpha X_2. \qquad (3)$$

Then, using (2) and (3), we have

$$\begin{aligned} AP = A[X_1|X_2] &= [AX_1|AX_2] \\ &= [\alpha X_1|X_1 + \alpha X_2] \\ &= [X_1|X_2]\begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}. \end{aligned}$$

Hence

$$AP = P\begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$$

and hence

$$P^{-1}AP = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}.$$

8. The system of differential equations is equivalent to the single matrix equation $\dot{X} = AX$, where $A = \begin{bmatrix} 4 & -1 \\ 4 & 8 \end{bmatrix}$.

The characteristic polynomial of A is $\lambda^2 - 12\lambda + 36 = (\lambda - 6)^2$, so we can use the previous question with $\alpha = 6$. Let

$$B = A - 6I_2 = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}.$$

Then $BX_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, if $X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Also let $X_1 = BX_2$. Then if $P = [X_1|X_2]$, we have

$$P^{-1}AP = \begin{bmatrix} 6 & 1 \\ 0 & 6 \end{bmatrix}.$$

Now make the change of variables $X = PY$, where $Y = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$. Then

$$\dot{Y} = (P^{-1}AP)Y = \begin{bmatrix} 6 & 1 \\ 0 & 6 \end{bmatrix} Y,$$

or equivalently $\dot{x_1} = 6x_1 + y_1$ and $\dot{y_1} = 6y_1$.

Solving for y_1 gives $y_1 = y_1(0)e^{6t}$. Consequently

$$\dot{x_1} = 6x_1 + y_1(0)e^{6t}.$$

Multiplying both side of this equation by e^{-6t} gives

$$\begin{aligned} \frac{d}{dt}(e^{-6t}x_1) &= e^{-6t}\dot{x_1} - 6e^{-6t}x_1 = y_1(0) \\ e^{-6t}x_1 &= y_1(0)t + c, \end{aligned}$$

where c is a constant. Substituting $t = 0$ gives $c = x_1(0)$. Hence

$$e^{-6t}x_1 = y_1(0)t + x_1(0)$$

and hence

$$x_1 = e^{6t}(y_1(0)t + x_1(0)).$$

However, since we are assuming $x(0) = 1 = y(0)$, we have

$$\begin{aligned}
\begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix} &= P^{-1} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \\
&= \frac{1}{-4} \begin{bmatrix} 0 & -1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 \\ -6 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/2 \end{bmatrix}.
\end{aligned}$$

Hence $x_1 = e^{6t}(\frac{3}{2}t + \frac{1}{4})$ and $y_1 = \frac{3}{2}e^{6t}$.

Finally, solving for x and y,

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -2 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} e^{6t}(\frac{3}{2}t + \frac{1}{4}) \\ \\ \frac{3}{2}e^{6t} \end{bmatrix} \\
&= \begin{bmatrix} (-2)e^{6t}(\frac{3}{2}t + \frac{1}{4}) + \frac{3}{2}e^{6t} \\ \\ 4e^{6t}(\frac{3}{2}t + \frac{1}{4}) \end{bmatrix} \\
&= \begin{bmatrix} e^{6t}(1 - 3t) \\ e^{6t}(6t + 1) \end{bmatrix}.
\end{aligned}$$

Hence $x = e^{6t}(1 - 3t)$ and $y = e^{6t}(6t + 1)$.

9. Let

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}.$$

(a) We first determine the characteristic polynomial $\mathrm{ch}_A(\lambda)$.

$$\begin{aligned}
\mathrm{ch}_A(\lambda) &= \det(\lambda I_3 - A) = \begin{vmatrix} \lambda - 1/2 & -1/2 & 0 \\ -1/4 & \lambda - 1/4 & -1/2 \\ -1/4 & -1/4 & \lambda - 1/2 \end{vmatrix} \\
&= \left(\lambda - \frac{1}{2}\right) \begin{vmatrix} \lambda - 1/4 & -1/2 \\ -1/4 & \lambda - 1/2 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} -1/4 & -1/2 \\ -1/4 & \lambda - 1/2 \end{vmatrix} \\
&= \left(\lambda - \frac{1}{2}\right) \left\{ \left(\lambda - \frac{1}{4}\right) \left(\lambda - \frac{1}{2}\right) - \frac{1}{8} \right\} + \frac{1}{2} \left\{ \frac{-1}{4} \left(\lambda - \frac{1}{2}\right) - \frac{1}{8} \right\} \\
&= \left(\lambda - \frac{1}{2}\right) \left(\lambda^2 - \frac{3\lambda}{4}\right) - \frac{\lambda}{8} \\
&= \lambda \left\{ \left(\lambda - \frac{1}{2}\right) \left(\lambda - \frac{3}{4}\right) - \frac{1}{8} \right\}
\end{aligned}$$

$$\begin{aligned} &= \lambda\left(\lambda^2 - \frac{5\lambda}{4} + \frac{1}{4}\right) \\ &= \lambda(\lambda - 1)\left(\lambda - \frac{1}{4}\right). \end{aligned}$$

(b) Hence the characteristic polynomial has no repeated roots and we can use Theorem 6.2.2 to find a non–singular matrix P such that

$$P^{-1}AP = \text{diag}(1,\, 0,\, \frac{1}{4}).$$

We take $P = [X_1|X_2|X_3]$, where X_1, X_2, X_3 are eigenvectors corresponding to the respective eigenvalues 1, 0, $\frac{1}{4}$.
Finding X_1: We have to solve $(A - I_3)X = 0$. we have

$$A - I_3 = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 1/4 & -3/4 & 1/2 \\ 1/4 & 1/4 & -1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the eigenspace consists of vectors $X = [x,\, y,\, z]^t$ satisfying $x = z$ and $y = z$, with z arbitrary. Hence

$$X = \begin{bmatrix} z \\ z \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and we can take $X_1 = [1,\, 1,\, 1]^t$.
Finding X_2: We solve $AX = 0$. We have

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the eigenspace consists of vectors $X = [x,\, y,\, z]^t$ satisfying $x = -y$ and $z = 0$, with y arbitrary. Hence

$$X = \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and we can take $X_2 = [-1,\, 1,\, 0]^t$.
Finding X_3: We solve $(A - \frac{1}{4}I_3)X = 0$. We have

$$A - \frac{1}{4}I_3 = \begin{bmatrix} 1/4 & 1/2 & 0 \\ 1/4 & 0 & 1/2 \\ 1/4 & 1/4 & 1/4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the eigenspace consists of vectors $X = [x,\ y,\ z]^t$ satisfying $x = -2z$ and $y = z$, with z arbitrary. Hence

$$X = \begin{bmatrix} -2z \\ z \\ 0 \end{bmatrix} = z \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

and we can take $X_3 = [-2,\ 1,\ 1]^t$.

Hence we can take $P = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

(c) $A = P\mathrm{diag}(1,\ 0,\ \frac{1}{4})P^{-1}$ so $A^n = P\mathrm{diag}(1,\ 0,\ \frac{1}{4^n})P^{-1}$.
Hence

$$\begin{aligned} A^n &= \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4^n} \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -3 \\ -1 & -1 & 2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & 0 & -\frac{2}{4^n} \\ 1 & 0 & \frac{1}{4^n} \\ 1 & 0 & \frac{1}{4^n} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -3 \\ -1 & -1 & 2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1+\frac{2}{4^n} & 1+\frac{2}{4^n} & 1-\frac{4}{4^n} \\ 1-\frac{1}{4^n} & 1-\frac{1}{4^n} & 1+\frac{2}{4^n} \\ 1-\frac{1}{4^n} & 1-\frac{1}{4^n} & 1+\frac{2}{4^n} \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{1}{3 \cdot 4^n} \begin{bmatrix} 2 & 2 & -4 \\ -1 & -1 & 2 \\ -1 & -1 & 2 \end{bmatrix}. \end{aligned}$$

10. Let

$$A = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 5 & -2 \\ -2 & -2 & 5 \end{bmatrix}.$$

(a) We first determine the characteristic polynomial $\mathrm{ch}_A(\lambda)$.

$$\begin{aligned} \mathrm{ch}_A(\lambda) &= \left| \begin{matrix} \lambda-5 & -2 & 2 \\ -2 & \lambda-5 & 2 \\ 2 & 2 & \lambda-5 \end{matrix} \right] \begin{matrix} R_3 \to R_3 + R_2 \\ = \end{matrix} \left| \begin{matrix} \lambda-5 & -2 & 2 \\ -2 & \lambda-5 & 2 \\ 0 & \lambda-3 & \lambda-3 \end{matrix} \right| \\ &= (\lambda-3) \left| \begin{matrix} \lambda-5 & -2 & 2 \\ -2 & \lambda-5 & 2 \\ 0 & 1 & 1 \end{matrix} \right| \end{aligned}$$

$$\begin{aligned}
C_3 \to C_3 - C_2 \quad &= (\lambda-3)\begin{vmatrix} \lambda-5 & -2 & 4 \\ -2 & \lambda-5 & -\lambda+7 \\ 0 & 1 & 0 \end{vmatrix} \\
&= -(\lambda-3)\begin{vmatrix} \lambda-5 & 4 \\ -2 & -\lambda+7 \end{vmatrix} \\
&= -(\lambda-3)\{(\lambda-5)(-\lambda+7)+8\} \\
&= -(\lambda-3)(-\lambda^2+5\lambda+7\lambda-35+8) \\
&= -(\lambda-3)(-\lambda^2+12\lambda-27) \\
&= -(\lambda-3)(-1)(\lambda-3)(\lambda-9) \\
&= (\lambda-3)^2(\lambda-9).
\end{aligned}$$

We have to find bases for each of the eigenspaces $N(A-9I_3)$ and $N(A-3I_3)$.

First we solve $(A-3I_3)X=0$. We have

$$A - 3I_3 = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the eigenspace consists of vectors $X = [x,\, y,\, z]^t$ satisfying $x = -y+z$, with y and z arbitrary. Hence

$$X = \begin{bmatrix} -y+z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

so $X_1 = [-1,\, 1,\, 0]^t$ and $X_2 = [1,\, 0,\, 1]^t$ form a basis for the eigenspace corresponding to the eigenvalue 3.

Next we solve $(A-9I_3)X=0$. We have

$$A - 9I_3 = \begin{bmatrix} -4 & 2 & -2 \\ 2 & -4 & -2 \\ -2 & -2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the eigenspace consists of vectors $X = [x,\, y,\, z]^t$ satisfying $x = -z$ and $y = -z$, with z arbitrary. Hence

$$X = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

and we can take $X_3 = [-1,\, -1,\, 1]^t$ as a basis for the eigenspace corresponding to the eigenvalue 9.

Then Theorem 6.2.3 assures us that $P = [X_1|X_2|X_3]$ is non–singular and

$$P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$