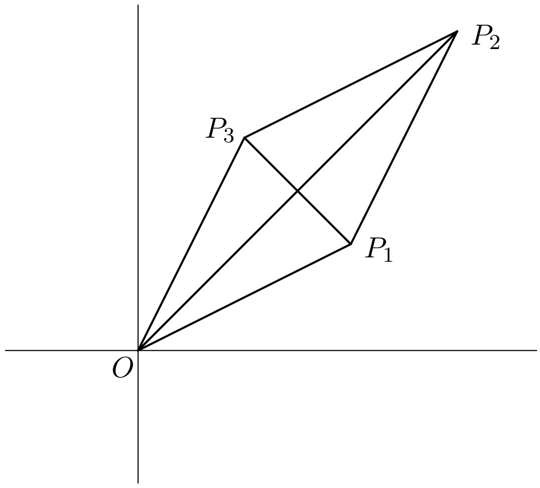


Section 4.1

1. We first prove that the area of a triangle $P_1P_2P_3$, where the points are in anti–clockwise orientation, is given by the formula

$$\frac{1}{2}\left\{\left|\begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array}\right| + \left|\begin{array}{cc} x_2 & x_3 \\ y_2 & y_3 \end{array}\right| + \left|\begin{array}{cc} x_3 & x_1 \\ y_3 & y_1 \end{array}\right|\right\}.$$

Referring to the above diagram, we have

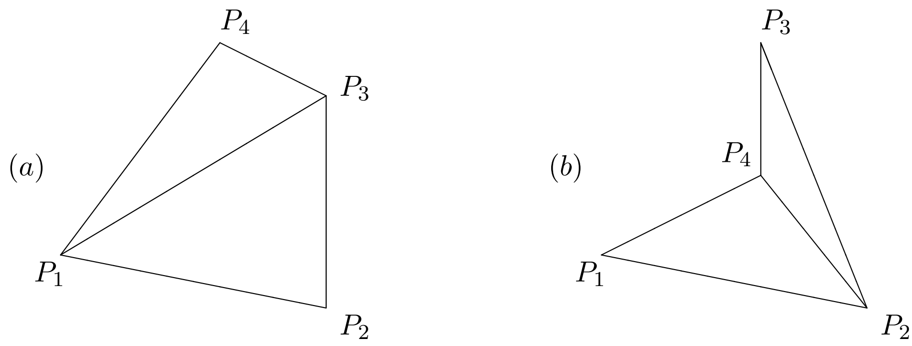
$$\begin{aligned} \text{Area}\, P_1P_2P_3 &= \text{Area}\, OP_1P_2 + \text{Area}\, OP_2P_3 - \text{Area}\, OP_1P_3 \\ &= \frac{1}{2}\left|\begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array}\right| + \frac{1}{2}\left|\begin{array}{cc} x_2 & x_3 \\ y_2 & y_3 \end{array}\right| - \frac{1}{2}\left|\begin{array}{cc} x_1 & x_3 \\ y_1 & y_3 \end{array}\right|, \end{aligned}$$

which gives the desired formula.

We now turn to the area of a quadrilateral. One possible configuration occurs when the quadrilateral is convex as in figure (a) below. The interior diagonal breaks the quadrilateral into two triangles $P_1P_2P_3$ and $P_1P_3P_4$. Then

$$\text{Area}\, P_1P_2P_3P_4 = \text{Area}\, P_1P_2P_3 + \text{Area}\, P_1P_3P_4$$

$$= \frac{1}{2}\left\{\left|\begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array}\right| + \left|\begin{array}{cc} x_2 & x_3 \\ y_2 & y_3 \end{array}\right| + \left|\begin{array}{cc} x_3 & x_1 \\ y_3 & y_1 \end{array}\right|\right\}$$



$$
\begin{aligned}
&+\frac{1}{2}\left\{\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & x_1 \\ y_4 & y_1 \end{vmatrix}\right\} \\
= \;& \frac{1}{2}\left\{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & x_1 \\ y_4 & y_1 \end{vmatrix}\right\},
\end{aligned}
$$

after cancellation.

Another possible configuration for the quadrilateral occurs when it is not convex, as in figure (b). The interior diagonal P_2P_4 then gives two triangles $P_1P_2P_4$ and $P_2P_3P_4$ and we can proceed similarly as before.

2.

$$
\Delta = \begin{vmatrix} a+x & b+y & c+z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} a & b & c \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} + \begin{vmatrix} x & y & z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix}.
$$

Now

$$
\begin{vmatrix} a & b & c \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ u+a & v+b & w+c \end{vmatrix} + \begin{vmatrix} a & b & c \\ u & v & w \\ u+a & v+b & w+c \end{vmatrix}
$$

$$
\begin{aligned}
&= \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} + \begin{vmatrix} a & b & c \\ x & y & z \\ a & b & c \end{vmatrix} + \begin{vmatrix} a & b & c \\ u & v & w \\ u & v & w \end{vmatrix} + \begin{vmatrix} a & b & c \\ u & v & w \\ a & b & c \end{vmatrix} \\
&= \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}.
\end{aligned}
$$

Similarly

$$
\begin{vmatrix} x & y & z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} x & y & z \\ u & v & w \\ a & b & c \end{vmatrix} = -\begin{vmatrix} x & y & z \\ a & b & c \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}.
$$

Hence $\Delta = 2\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}$.

3. $$\begin{vmatrix} n^2 & (n+1)^2 & (n+2)^2 \\ (n+1)^2 & (n+2)^2 & (n+3)^2 \\ (n+2)^2 & (n+3)^2 & (n+4)^2 \end{vmatrix} \begin{matrix} C_3 \to C_3 - C_2 \\ C_2 \to C_2 - C_1 \\ = \end{matrix} \begin{vmatrix} n^2 & 2n+1 & 2n+3 \\ (n+1)^2 & 2n+3 & 2n+5 \\ (n+2)^2 & 2n+5 & 2n+7 \end{vmatrix}$$

$$\begin{matrix} C_3 \to C_3 - C_2 \\ = \end{matrix} \begin{vmatrix} n^2 & 2n+1 & 2 \\ (n+1)^2 & 2n+3 & 2 \\ (n+2)^2 & 2n+5 & 2 \end{vmatrix}$$

$$\begin{matrix} R_3 \to R_3 - R_2 \\ R_2 \to R_2 - R_1 \\ = \end{matrix} \begin{vmatrix} n^2 & 2n+1 & 2 \\ 2n+1 & 2 & 0 \\ 2n+3 & 2 & 0 \end{vmatrix} = -8.$$

4. (a)

$$\begin{vmatrix} 246 & 427 & 327 \\ 1014 & 543 & 443 \\ -342 & 721 & 621 \end{vmatrix} = \begin{vmatrix} 246 & 100 & 327 \\ 1014 & 100 & 443 \\ -342 & 100 & 621 \end{vmatrix} = 100\begin{vmatrix} 246 & 1 & 327 \\ 1014 & 1 & 443 \\ -342 & 1 & 621 \end{vmatrix}$$

$$= \quad 100\begin{vmatrix} 246 & 1 & 327 \\ 768 & 0 & 116 \\ -588 & 0 & 294 \end{vmatrix} = 100(-1)\begin{vmatrix} 768 & 116 \\ -588 & 294 \end{vmatrix} = -29400000.$$

(b)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 2 & 11 \\ 0 & -10 & -10 & -10 \\ 0 & -5 & -14 & -17 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & 2 & 11 \\ -10 & -10 & -10 \\ -5 & -14 & -17 \end{vmatrix} = -10\begin{vmatrix} 5 & 2 & 11 \\ 1 & 1 & 1 \\ -5 & -14 & -17 \end{vmatrix}$$

$$= \quad -10\begin{vmatrix} 5 & -3 & 6 \\ 1 & 0 & 0 \\ -5 & -9 & -12 \end{vmatrix} = -10(-1)\begin{vmatrix} -3 & 6 \\ -9 & -12 \end{vmatrix} = 900.$$

5. $\det A = \begin{vmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \\ 5 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 10 \\ 5 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 10 \\ 2 & 7 \end{vmatrix} = -13.$

Hence A is non–singular and

$$A^{-1} = \frac{1}{-13}\,\mathrm{adj}\,A = \frac{1}{-13}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \frac{1}{-13}\begin{bmatrix} -11 & -4 & 2 \\ 29 & 7 & -10 \\ 1 & -2 & 1 \end{bmatrix}.$$

6. (i)

$$\begin{vmatrix} 2a & 2b & b-c \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix} \overset{R_1 \to R_1 + R_2}{=} \begin{vmatrix} 2a+2b & 2b+2a & b+a \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix}$$

$$= (a{+}b)\begin{vmatrix} 2 & 2 & 1 \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix} \overset{C_1 \to C_1 - C_2}{=} (a{+}b)\begin{vmatrix} 0 & 2 & 1 \\ 2(b-a) & 2a & a+c \\ 0 & a+b & b \end{vmatrix}$$

$$= 2(a+b)(a-b)\begin{vmatrix} 2 & 1 \\ a+b & b \end{vmatrix} = -2(a+b)(a-b)^2.$$

(ii)

$$\begin{vmatrix} b+c & b & c \\ c & c+a & a \\ b & a & a+b \end{vmatrix} \overset{C_1 \to C_1 - C_2}{=} \begin{vmatrix} c & b & c \\ -a & c+a & a \\ b-a & a & a+b \end{vmatrix}$$

$$\overset{C_3 \to C_3 - C_1}{=} \begin{vmatrix} c & b & 0 \\ -a & c+a & 2a \\ b-a & a & 2a \end{vmatrix} = 2a\begin{vmatrix} c & b & 0 \\ -a & c+a & 1 \\ b-a & a & 1 \end{vmatrix}$$

$$\overset{R_3 \to R_3 - R_2}{=} 2a\begin{vmatrix} c & b & 0 \\ -a & c+a & 1 \\ b & -c & 0 \end{vmatrix} = -2a\begin{vmatrix} c & b \\ b & -c \end{vmatrix} = 2a(c^2+b^2).$$

7. Suppose that the curve $y = ax^2 + bx + c$ passes through the points (x_1, y_1), (x_2, y_2), (x_3, y_3), where $x_i \neq x_j$ if $i \neq j$. Then

$$\begin{aligned} ax_1^2 + bx_1 + c &= y_1 \\ ax_2^2 + bx_2 + c &= y_2 \\ ax_3^2 + bx_3 + c &= y_3. \end{aligned}$$

The coefficient determinant is essentially a Vandermonde determinant:

$$\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = -(x_2-x_1)(x_3-x_1)(x_3-x_2).$$

Hence the coefficient determinant is non–zero and by Cramer's rule, there is a unique solution for a, b, c.

8. Let $\Delta = \det A = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{vmatrix}$. Then

$$\begin{aligned} \Delta = \quad & \begin{matrix} C_3 \to C_3 + C_1 \\ C_2 \to C_2 - C_1 \end{matrix} \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & k+2 \\ 1 & k-1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & k+2 \\ k-1 & 4 \end{vmatrix} \\ = \quad & 4 - (k-1)(k+2) = -(k^2 - k - 6) = -(k+3)(k-2). \end{aligned}$$

Hence $\det A = 0$ if and only if $k = -3$ or $k = 2$.

Consequently if $k \neq -3$ and $k \neq 2$, then $\det A \neq 0$ and the given system

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + kz &= 3 \\ x + ky + 3z &= 2 \end{aligned}$$

has a unique solution. We consider the cases $k = -3$ and $k = 2$ separately.
$k = -3$:

$$AM = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & -3 & 3 \\ 1 & -3 & 3 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -4 & 4 & 1 \end{bmatrix}$$

$$R_3 \to R_3 + 4R_2 \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

from which we read off inconsistency.
$k = 2$:

$$AM = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 2 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 4 & 1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We read off the complete solution $x = 5z$, $y = 1 - 4z$, where z is arbitrary.

Finally we have to determine the solution for which $x^2+y^2+z^2$ is least.

$$\begin{aligned} x^2+y^2+z^2 &= (5z)^2+(1-4z)^2+z^2 = 42z^2-8z+1 \\ &= 42(z^2-\frac{4}{21}z+\frac{1}{42}) = 42\left\{\left(z-\frac{2}{21}\right)^2+\frac{1}{42}-\left(\frac{2}{21}\right)^2\right\} \\ &= 42\left\{\left(z-\frac{2}{21}\right)^2+\frac{13}{882}\right\}. \end{aligned}$$

We see that the least value of $x^2+y^2+z^2$ is $42\times\frac{13}{882}=\frac{13}{21}$ and this occurs when $z=2/21$, with corresponding values $x=10/21$ and $y=1-4\times\frac{2}{21}=13/21$.

9. Let $\Delta = \left[\begin{array}{rrr|} 1 & -2 & b \\ a & 0 & 2 \\ 5 & 2 & 0 \end{array}\right.$ be the coefficient determinant of the given system. Then expanding along column 2 gives

$$\begin{aligned} \Delta &= 2\begin{vmatrix} a & 2 \\ 5 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & b \\ a & 2 \end{vmatrix} = -20-2(2-ab) \\ &= 2ab-24 = 2(ab-12). \end{aligned}$$

Hence $\Delta=0$ if and only if $ab=12$. Hence if $ab\neq 12$, the given system has a unique solution.

If $ab=12$ we must argue with care:

$$\begin{aligned} AM &= \begin{bmatrix} 1 & -2 & b & 3 \\ a & 0 & 2 & 2 \\ 5 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & b & 3 \\ 0 & 2a & 2-ab & 2-3a \\ 0 & 12 & -5b & -14 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & -2 & b & 3 \\ 0 & 1 & \frac{-5b}{12} & \frac{-7}{6} \\ 0 & 2a & 2-ab & 2-3a \end{bmatrix} \to \begin{bmatrix} 1 & -2 & b & 3 \\ 0 & 1 & \frac{-5b}{12} & \frac{-7}{6} \\ 0 & 0 & \frac{12-ab}{6} & \frac{6-2a}{3} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & b & 3 \\ 0 & 1 & \frac{-5b}{12} & \frac{-7}{6} \\ 0 & 0 & 0 & \frac{6-2a}{3} \end{bmatrix} = B. \end{aligned}$$

Hence if $6-2a\neq 0$, i.e. $a\neq 3$, the system has no solution.

If $a=3$ (and hence $b=4$), then

$$B = \begin{bmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & \frac{-5}{3} & \frac{-7}{6} \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2/3 & 2/3 \\ 0 & 1 & \frac{-5}{3} & \frac{-7}{6} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consequently the complete solution of the system is $x = -\frac{2}{3} + \frac{2}{3}z,\ y = \frac{-7}{6} + \frac{5}{3}z$, where z is arbitrary. Hence there are infinitely many solutions.

10.

$$\Delta = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 2t+6 \\ 2 & 2 & 6-t & t \end{vmatrix} \begin{matrix} R_4 \to R_4 - 2R_1 \\ R_3 \to R_3 - 2R_1 \\ R_2 \to R_2 - R_1 \\ = \end{matrix} \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 2t+4 \\ 0 & 0 & 2-t & t-2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & 2t+4 \\ 0 & 2-t & t-2 \end{vmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ = \end{matrix} \begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 2t-2 \\ 0 & 2-t & t-2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2t-2 \\ 2-t & t-2 \end{vmatrix} = (t-2)\begin{vmatrix} 1 & 2t-2 \\ -1 & 1 \end{vmatrix} = (t-2)(2t-1).$$

Hence $\Delta = 0$ if and only if $t = 2$ or $t = \frac{1}{2}$. Consequently the given matrix B is non–singular if and only if $t \neq 2$ and $t \neq \frac{1}{2}$.

11. Let A be a 3×3 matrix with $\det A \neq 0$. Then

(i)

$$\begin{aligned} A \operatorname{adj} A &= (\det A) I_3 \qquad (1) \\ (\det A) \det(\operatorname{adj} A) &= \det(\det A \cdot I_3) = (\det A)^3. \end{aligned}$$

Hence, as $\det A \neq 0$, dividing out by $\det A$ in the last equation gives

$$\det(\operatorname{adj} A) = (\det A)^2.$$

(ii) . Also from equation (1)

$$\left(\frac{1}{\det A} A\right) \operatorname{adj} A = I_3,$$

so $\operatorname{adj} A$ is non–singular and

$$(\operatorname{adj} A)^{-1} = \frac{1}{\det A} A.$$

Finally

$$A^{-1} \operatorname{adj}(A^{-1}) = (\det A^{-1}) I_3$$

and multiplying both sides of the last equation by A gives

$$\operatorname{adj}(A^{-1}) = A(\det A^{-1}) I_3 = \frac{1}{\det A} A.$$

12. Let A be a real 3×3 matrix satisfying $A^tA = I_3$. Then

$$\begin{aligned} \text{(i) } A^t(A - I_3) &= A^tA - A^t = I_3 - A^t \\ &= -(A^t - I_3) = -(A^t - I_3^t) = -(A - I_3)^t. \end{aligned}$$

Taking determinants of both sides then gives

$$\begin{aligned} \det A^t \det(A - I_3) &= \det(-(A - I_3)^t) \\ \det A \det(A - I_3) &= (-1)^3 \det(A - I_3)^t \\ &= -\det(A - I_3) \qquad (1). \end{aligned}$$

(ii) Also $\det AA^t = \det I_3$, so

$$\det A^t \det A = 1 = (\det A)^2.$$

Hence $\det A = \pm 1$.
(iii) Suppose that $\det A = 1$. Then equation (1) gives

$$\det(A - I_3) = -\det(A - I_3),$$

so $(1 + 1)\det(A - I_3) = 0$ and hence $\det(A - I_3) = 0$.

13. Suppose that column 1 is a linear combination of the remaining columns:

$$A_{*1} = x_2A_{*2} + \cdots + x_nA_{*n}.$$

Then

$$\det A = \begin{vmatrix} x_2a_{12} + \cdots + x_na_{1n} & a_{12} & \cdots & a_{1n} \\ x_2a_{22} + \cdots + x_na_{2n} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_2a_{n2} + \cdots + x_na_{nn} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Now $\det A$ is unchanged in value if we perform the operation

$$C_1 \to C_1 - x_2C_2 - \cdots - x_nC_n :$$

$$\det A = \begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0.$$

Conversely, suppose that $\det A = 0$. Then the homogeneous system $AX = 0$ has a non–trivial solution $X = [x_1, \ldots, x_n]^t$. So

$$x_1 A_{*1} + \cdots + x_n A_{*n} = 0.$$

Suppose for example that $x_1 \neq 0$. Then

$$A_{*1} = \left(-\frac{x_2}{x_1}\right) + \cdots + \left(-\frac{x_n}{x_1}\right) A_{*n}$$

and the first column of A is a linear combination of the remaining columns.

14. Consider the system

$$\begin{array}{rcr} -2x + 3y - z &=& 1 \\ x + 2y - z &=& 4 \\ -2x - y + z &=& -3 \end{array}$$

Let $\Delta = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 7 & -3 \\ 1 & 2 & -1 \\ 0 & 3 & -1 \end{vmatrix} = -\begin{vmatrix} 7 & -3 \\ 3 & -1 \end{vmatrix} = -2 \neq 0.$

Hence the system has a unique solution which can be calculated using Cramer's rule:

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta},$$

where

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix} = -4, \\ \Delta_2 &= \begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix} = -6, \\ \Delta_3 &= \begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix} = -8. \end{aligned}$$

Hence $x = \frac{-4}{-2} = 2$, $y = \frac{-6}{-2} = 3$, $z = \frac{-8}{-2} = 4$.

15. In Remark 4.0.4, take $A = I_n$. Then we deduce

(a) $\det E_{ij} = -1$;

(b) $\det E_i(t) = t$;

(c) $\det E_{ij}(t) = 1$.

Now suppose that B is a non–singular $n \times n$ matrix. Then we know that B is a product of elementary row matrices:

$$B = E_1 \cdots E_m.$$

Consequently we have to prove that

$$\det E_1 \cdots E_m A = \det E_1 \cdots E_m \det A.$$

We prove this by induction on m.

First the case $m = 1$. We have to prove $\det E_1 A = \det E_1 \det A$ if E_1 is an elementary row matrix. This follows form Remark 4.0.4:

(a) $\det E_{ij} A = -\det A = \det E_{ij} \det A$;

(b) $\det E_i(t) A = t \det A = \det E_i(t) \det A$;

(c) $\det E_{ij}(t) A = \det A = \det E_{ij}(t) \det A$.

Let $m \geq 1$ and assume the proposition holds for products of m elementary row matrices. Then

$$\begin{aligned}
\det E_1 \cdots E_m E_{m+1} A &= \det (E_1 \cdots E_m)(E_{m+1} A) \\
&= \det (E_1 \cdots E_m) \det (E_{m+1} A) \\
&= \det (E_1 \cdots E_m) \det E_{m+1} \det A \\
&= \det ((E_1 \cdots E_m) E_{m+1}) \det A
\end{aligned}$$

and the induction goes through.

Hence $\det BA = \det B \det A$ if B is non–singular.

If B is singular, problem 26, Chapter 2.7 tells us that BA is also singlular. However singular matrices have zero determinant, so

$$\det B = 0 \quad \det BA = 0,$$

so the equation $\det BA = \det B \det A$ holds trivially in this case.

16.

$$\begin{vmatrix}
a+b+c & a+b & a & a \\
a+b & a+b+c & a & a \\
a & a & a+b+c & a+b \\
a & a & a+b & a+b+c
\end{vmatrix}$$

$$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3,\ R_3 \to R_3 - R_4 \; = \begin{vmatrix} c & -c & 0 & 0 \\ b & b+c & -b-c & -b \\ 0 & 0 & c & -c \\ a & a & a+b & a+b+c \end{vmatrix}$$

$$C_2 \to C_2 + C_1 \; = \begin{vmatrix} c & 0 & 0 & 0 \\ b & 2b+c & -b-c & -b \\ 0 & 0 & c & -c \\ a & 2a & a+b & a+b+c \end{vmatrix} = c \begin{vmatrix} 2b+c & -b-c & -b \\ 0 & c & -c \\ 2a & a+b & a+b+c \end{vmatrix}$$

$$C_3 \to C_3 + C_2 \; = c \begin{vmatrix} 2b+c & -b-c & -2b-c \\ 0 & c & 0 \\ 2a & a+b & 2a+2b+c \end{vmatrix} = c^2 \begin{vmatrix} 2b+c & -2b-c \\ 2a & 2a+2b+c \end{vmatrix}$$

$$= c^2(2b+c) \begin{vmatrix} 1 & -1 \\ 2a & 2a+2b+c \end{vmatrix} = c^2(2b+c)(4a+2b+c).$$

17. Let $\Delta = \begin{vmatrix} 1+u_1 & u_1 & u_1 & u_1 \\ u_2 & 1+u_2 & u_2 & u_2 \\ u_3 & u_3 & 1+u_3 & u_3 \\ u_4 & u_4 & u_4 & 1+u_4 \end{vmatrix}$. Then using the operation

$$R_1 \to R_1 + R_2 + R_3 + R_4$$

we have

$$\Delta = \begin{vmatrix} t & t & t & t \\ u_2 & 1+u_2 & u_2 & u_2 \\ u_3 & u_3 & 1+u_3 & u_3 \\ u_4 & u_4 & u_4 & 1+u_4 \end{vmatrix}$$

(where $t = 1 + u_1 + u_2 + u_3 + u_4$)

$$= (1 + u_1 + u_2 + u_3 + u_4) \begin{vmatrix} 1 & 1 & 1 & 1 \\ u_2 & 1+u_2 & u_2 & u_2 \\ u_3 & u_3 & 1+u_3 & u_3 \\ u_4 & u_4 & u_4 & 1+u_4 \end{vmatrix}$$

The last determinant equals

$$C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1,\ C_4 \to C_4 - C_1 \; \begin{vmatrix} 1 & 0 & 0 & 0 \\ u_2 & 1 & 0 & 0 \\ u_3 & 0 & 1 & 0 \\ u_4 & 0 & 0 & 1 \end{vmatrix} = 1.$$

18. Suppose that $A^t = -A$, that $A \in M_{n\times n}(F)$, where n is odd. Then

$$\begin{aligned} \det A^t &= \det(-A) \\ \det A &= (-1)^n \det A = -\det A. \end{aligned}$$

Hence $(1+1)\det A = 0$ and consequently $\det A = 0$ if $1+1 \neq 0$ in F.

19.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \\ r & r & 1 & 1 \\ r & r & r & 1 \end{vmatrix} = \begin{matrix} C_4 \to C_4 - C_3 \\ C_3 \to C_3 - C_2 \\ C_2 \to C_2 - C_1 \\ = \end{matrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ r & 1-r & 0 & 0 \\ r & 0 & 1-r & 0 \\ r & 0 & 0 & 1-r \end{vmatrix} = (1-r)^3.$$

20.

$$\begin{vmatrix} 1 & a^2-bc & a^4 \\ 1 & b^2-ca & b^4 \\ 1 & c^2-ab & c^4 \end{vmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ = \end{matrix} \begin{vmatrix} 1 & a^2-bc & a^4 \\ 0 & b^2-ca-a^2+bc & b^4-a^4 \\ 0 & c^2-ab-a^2+bc & c^4-a^4 \end{vmatrix}$$

$$\begin{aligned} &= \begin{vmatrix} b^2-ca-a^2+bc & b^4-a^4 \\ c^2-ab-a^2+bc & c^4-a^4 \end{vmatrix} \\ &= \begin{vmatrix} (b-a)(b+a)+c(b-a) & (b-a)(b+a)(b^2+a^2) \\ (c-a)(c+a)+b(c-a) & (c-a)(c+a)(c^2+a^2) \end{vmatrix} \\ &= \begin{vmatrix} (b-a)(b+a+c) & (b-a)(b+a)(b^2+a^2) \\ (c-a)(c+a+b) & (c-a)(c+a)(c^2+a^2) \end{vmatrix} \\ &= (b-a)(c-a)\begin{vmatrix} b+a+c & (b+a)(b^2+a^2) \\ c+a+b & (c+a)(c^2+a^2) \end{vmatrix} \\ &= (b-a)(c-a)(a+b+c)\begin{vmatrix} 1 & (b+a)(b^2+a^2) \\ 1 & (c+a)(c^2+a^2) \end{vmatrix}. \end{aligned}$$

Finally

$$\begin{aligned} \begin{vmatrix} 1 & (b+a)(b^2+a^2) \\ 1 & (c+a)(c^2+a^2) \end{vmatrix} &= (c^3+ac^2+ca^2+a^3)-(b^3+ab^2+ba^2+a^3) \\ &= (c^3-b^3)+a(c^2-b^2)+a^2(c-b) \\ &= (c-b)(c^2+cb+b^2+a(c+b)+a^2) \\ &= (c-b)(c^2+cb+b^2+ac+ab+a^2). \end{aligned}$$