

Section 4.1

1. We first prove that the area of a triangle $P_1P_2P_3$, where the points are in anti-clockwise orientation, is given by the formula

$$\frac{1}{2} \left\{ \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix} \right\}.$$

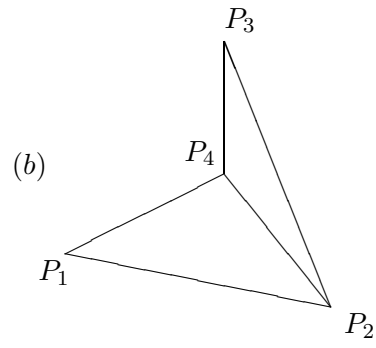
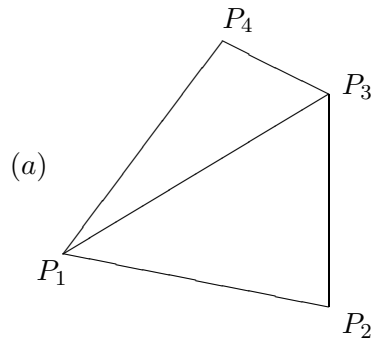
Referring to the above diagram, we have

$$\begin{aligned} \text{Area } P_1P_2P_3 &= \text{Area } OP_1P_2 + \text{Area } OP_2P_3 - \text{Area } OP_1P_3 \\ &= \frac{1}{2} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \frac{1}{2} \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \end{aligned}$$

which gives the desired formula.

We now turn to the area of a quadrilateral. One possible configuration occurs when the quadrilateral is convex as in figure (a) below. The interior diagonal breaks the quadrilateral into two triangles $P_1P_2P_3$ and $P_1P_3P_4$. Then

$$\begin{aligned} \text{Area } P_1P_2P_3P_4 &= \text{Area } P_1P_2P_3 + \text{Area } P_1P_3P_4 \\ &= \frac{1}{2} \left\{ \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix} \right\} \end{aligned}$$



$$\begin{aligned}
& + \frac{1}{2} \left\{ \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & x_1 \\ y_4 & y_1 \end{vmatrix} \right\} \\
= & \frac{1}{2} \left\{ \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & x_1 \\ y_4 & y_1 \end{vmatrix} \right\},
\end{aligned}$$

after cancellation.

Another possible configuration for the quadrilateral occurs when it is not convex, as in figure (b). The interior diagonal P_2P_4 then gives two triangles $P_1P_2P_4$ and $P_2P_3P_4$ and we can proceed similarly as before.

2.

$$\Delta = \begin{vmatrix} a+x & b+y & c+z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} a & b & c \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} + \begin{vmatrix} x & y & z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix}.$$

Now

$$\begin{aligned}
\begin{vmatrix} a & b & c \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} &= \begin{vmatrix} a & b & c \\ x & y & z \\ u+a & v+b & w+c \end{vmatrix} + \begin{vmatrix} a & b & c \\ u & v & w \\ u+a & v+b & w+c \end{vmatrix} \\
&= \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} + \begin{vmatrix} a & b & c \\ x & y & z \\ a & b & c \end{vmatrix} + \begin{vmatrix} a & b & c \\ u & v & w \\ u & v & w \end{vmatrix} + \begin{vmatrix} a & b & c \\ u & v & w \\ a & b & c \end{vmatrix} \\
&= \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}.
\end{aligned}$$

Similarly

$$\begin{vmatrix} x & y & z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} x & y & z \\ u & v & w \\ a & b & c \end{vmatrix} = - \begin{vmatrix} x & y & z \\ a & b & c \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}.$$

$$\text{Hence } \Delta = 2 \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}.$$

$$\begin{aligned} 3. \quad & \begin{vmatrix} n^2 & (n+1)^2 & (n+2)^2 \\ (n+1)^2 & (n+2)^2 & (n+3)^2 \\ (n+2)^2 & (n+3)^2 & (n+4)^2 \end{vmatrix} \begin{array}{l} C_3 \rightarrow C_3 - C_2 \\ C_2 \rightarrow C_2 - C_1 \\ = \end{array} \begin{vmatrix} n^2 & 2n+1 & 2n+3 \\ (n+1)^2 & 2n+3 & 2n+5 \\ (n+2)^2 & 2n+5 & 2n+7 \end{vmatrix} \\ & \begin{array}{l} C_3 \rightarrow C_3 - C_2 \\ = \end{array} \begin{vmatrix} n^2 & 2n+1 & 2 \\ (n+1)^2 & 2n+3 & 2 \\ (n+2)^2 & 2n+5 & 2 \end{vmatrix} \\ & \begin{array}{l} R_3 \rightarrow R_3 - R_2 \\ R_2 \rightarrow R_2 - R_1 \\ = \end{array} \begin{vmatrix} n^2 & 2n+1 & 2 \\ 2n+1 & 2 & 0 \\ 2n+3 & 2 & 0 \end{vmatrix} = -8. \end{aligned}$$

4. (a)

$$\begin{aligned} & \begin{vmatrix} 246 & 427 & 327 \\ 1014 & 543 & 443 \\ -342 & 721 & 621 \end{vmatrix} = \begin{vmatrix} 246 & 100 & 327 \\ 1014 & 100 & 443 \\ -342 & 100 & 621 \end{vmatrix} = 100 \begin{vmatrix} 246 & 1 & 327 \\ 1014 & 1 & 443 \\ -342 & 1 & 621 \end{vmatrix} \\ & = 100 \begin{vmatrix} 246 & 1 & 327 \\ 768 & 0 & 116 \\ -588 & 0 & 294 \end{vmatrix} = 100(-1) \begin{vmatrix} 768 & 116 \\ -588 & 294 \end{vmatrix} = -29400000. \end{aligned}$$

(b)

$$\begin{aligned} & \begin{vmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 2 & 11 \\ 0 & -10 & -10 & -10 \\ 0 & -5 & -14 & -17 \end{vmatrix} \\ & = \begin{vmatrix} 5 & 2 & 11 \\ -10 & -10 & -10 \\ -5 & -14 & -17 \end{vmatrix} = -10 \begin{vmatrix} 5 & 2 & 11 \\ 1 & 1 & 1 \\ -5 & -14 & -17 \end{vmatrix} \\ & = -10 \begin{vmatrix} 5 & -3 & 6 \\ 1 & 0 & 0 \\ -5 & -9 & -12 \end{vmatrix} = -10(-1) \begin{vmatrix} -3 & 6 \\ -9 & -12 \end{vmatrix} = 900. \end{aligned}$$

$$5. \det A = \begin{vmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \\ 5 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 10 \\ 5 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 10 \\ 2 & 7 \end{vmatrix} = -13.$$

Hence A is non-singular and

$$A^{-1} = \frac{1}{-13} \text{adj } A = \frac{1}{-13} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \frac{1}{-13} \begin{bmatrix} -11 & -4 & 2 \\ 29 & 7 & -10 \\ 1 & -2 & 1 \end{bmatrix}.$$

6. (i)

$$\begin{aligned} & \left| \begin{array}{ccc} 2a & 2b & b-c \\ 2b & 2a & a+c \\ a+b & a+b & b \end{array} \right| \begin{array}{l} R_1 \rightarrow R_1 + R_2 \\ = \end{array} \left| \begin{array}{ccc} 2a+2b & 2b+2a & b+a \\ 2b & 2a & a+c \\ a+b & a+b & b \end{array} \right| \\ &= (a+b) \left| \begin{array}{ccc} 2 & 2 & 1 \\ 2b & 2a & a+c \\ a+b & a+b & b \end{array} \right| \begin{array}{l} C_1 \rightarrow C_1 - C_2 \\ = \end{array} (a+b) \left| \begin{array}{ccc} 0 & 2 & 1 \\ 2(b-a) & 2a & a+c \\ 0 & a+b & b \end{array} \right| \\ &= 2(a+b)(a-b) \left| \begin{array}{cc} 2 & 1 \\ a+b & b \end{array} \right| = -2(a+b)(a-b)^2. \end{aligned}$$

(ii)

$$\begin{aligned} & \left| \begin{array}{ccc} b+c & b & c \\ c & c+a & a \\ b & a & a+b \end{array} \right| \begin{array}{l} C_1 \rightarrow C_1 - C_2 \\ = \end{array} \left| \begin{array}{ccc} c & b & c \\ -a & c+a & a \\ b-a & a & a+b \end{array} \right| \\ & \begin{array}{l} C_3 \rightarrow C_3 - C_1 \\ = \end{array} \left| \begin{array}{ccc} c & b & 0 \\ -a & c+a & 2a \\ b-a & a & 2a \end{array} \right| = 2a \left| \begin{array}{ccc} c & b & 0 \\ -a & c+a & 1 \\ b-a & a & 1 \end{array} \right| \\ & \begin{array}{l} R_3 \rightarrow R_3 - R_2 \\ = \end{array} 2a \left| \begin{array}{ccc} c & b & 0 \\ -a & c+a & 1 \\ b & -c & 0 \end{array} \right| = -2a \left| \begin{array}{cc} c & b \\ b & -c \end{array} \right| = 2a(c^2 + b^2). \end{aligned}$$

7. Suppose that the curve $y = ax^2 + bx + c$ passes through the points (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , where $x_i \neq x_j$ if $i \neq j$. Then

$$\begin{aligned} ax_1^2 + bx_1 + c &= y_1 \\ ax_2^2 + bx_2 + c &= y_2 \\ ax_3^2 + bx_3 + c &= y_3. \end{aligned}$$

The coefficient determinant is essentially a Vandermonde determinant:

$$\left| \begin{array}{ccc} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{array} \right| = \left| \begin{array}{ccc} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{array} \right| = - \left| \begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{array} \right| = -(x_2-x_1)(x_3-x_1)(x_3-x_2).$$

Hence the coefficient determinant is non-zero and by Cramer's rule, there is a unique solution for a, b, c .

8. Let $\Delta = \det A = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{vmatrix}$. Then

$$\begin{aligned} \Delta &= \begin{matrix} C_3 \rightarrow C_3 + C_1 \\ C_2 \rightarrow C_2 - C_1 \end{matrix} \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & k+2 \\ 1 & k-1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & k+2 \\ k-1 & 4 \end{vmatrix} \\ &= 4 - (k-1)(k+2) = -(k^2 - k - 6) = -(k+3)(k-2). \end{aligned}$$

Hence $\det A = 0$ if and only if $k = -3$ or $k = 2$.

Consequently if $k \neq -3$ and $k \neq 2$, then $\det A \neq 0$ and the given system

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + kz &= 3 \\ x + ky + 3z &= 2 \end{aligned}$$

has a unique solution. We consider the cases $k = -3$ and $k = 2$ separately.
 $k = -3$:

$$\begin{aligned} AM &= \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & -3 & 3 \\ 1 & -3 & 3 & 2 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -4 & 4 & 1 \end{bmatrix} \\ & \quad R_3 \rightarrow R_3 + 4R_2 \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \end{aligned}$$

from which we read off inconsistency.

$k = 2$:

$$\begin{aligned} AM &= \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 2 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 4 & 1 \end{bmatrix} \\ & \quad R_3 \rightarrow R_3 - R_2 \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

We read off the complete solution $x = 5z$, $y = 1 - 4z$, where z is arbitrary.

Finally we have to determine the solution for which $x^2 + y^2 + z^2$ is least.

$$\begin{aligned} x^2 + y^2 + z^2 &= (5z)^2 + (1 - 4z)^2 + z^2 = 42z^2 - 8z + 1 \\ &= 42\left(z^2 - \frac{4}{21}z + \frac{1}{42}\right) = 42\left\{\left(z - \frac{2}{21}\right)^2 + \frac{1}{42} - \left(\frac{2}{21}\right)^2\right\} \\ &= 42\left\{\left(z - \frac{2}{21}\right)^2 + \frac{13}{882}\right\}. \end{aligned}$$

We see that the least value of $x^2 + y^2 + z^2$ is $42 \times \frac{13}{882} = \frac{13}{21}$ and this occurs when $z = 2/21$, with corresponding values $x = 10/21$ and $y = 1 - 4 \times \frac{2}{21} = 13/21$.

9. Let $\Delta = \begin{vmatrix} 1 & -2 & b \\ a & 0 & 2 \\ 5 & 2 & 0 \end{vmatrix}$ be the coefficient determinant of the given system.

Then expanding along column 2 gives

$$\begin{aligned} \Delta &= 2 \begin{vmatrix} a & 2 \\ 5 & 0 \end{vmatrix} - 2 \begin{vmatrix} 1 & b \\ a & 2 \end{vmatrix} = -20 - 2(2 - ab) \\ &= 2ab - 24 = 2(ab - 12). \end{aligned}$$

Hence $\Delta = 0$ if and only if $ab = 12$. Hence if $ab \neq 12$, the given system has a unique solution.

If $ab = 12$ we must argue with care:

$$\begin{aligned} AM &= \begin{bmatrix} 1 & -2 & b & 3 \\ a & 0 & 2 & 2 \\ 5 & 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & b & 3 \\ 0 & 2a & 2 - ab & 2 - 3a \\ 0 & 12 & -5b & -14 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & -2 & b & 3 \\ 0 & 1 & \frac{-5b}{12} & \frac{-7}{6} \\ 0 & 2a & 2 - ab & 2 - 3a \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & b & 3 \\ 0 & 1 & \frac{-5b}{12} & \frac{-7}{6} \\ 0 & 0 & \frac{12-ab}{6} & \frac{6-2a}{3} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & b & 3 \\ 0 & 1 & \frac{-5b}{12} & \frac{-7}{6} \\ 0 & 0 & 0 & \frac{6-2a}{3} \end{bmatrix} = B. \end{aligned}$$

Hence if $6 - 2a \neq 0$, i.e. $a \neq 3$, the system has no solution.

If $a = 3$ (and hence $b = 4$), then

$$B = \begin{bmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & \frac{-5}{3} & \frac{-7}{6} \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2/3 & 2/3 \\ 0 & 1 & \frac{-5}{3} & \frac{-7}{6} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consequently the complete solution of the system is $x = \frac{2}{3} + \frac{2}{3}z$, $y = \frac{-7}{6} + \frac{5}{3}z$, where z is arbitrary. Hence there are infinitely many solutions.

10.

$$\begin{aligned} \Delta &= \left| \begin{array}{cccc|l} 1 & 1 & 2 & 1 & R_4 \rightarrow R_4 - 2R_1 \\ 1 & 2 & 3 & 4 & R_3 \rightarrow R_3 - 2R_1 \\ 2 & 4 & 7 & 2t+6 & R_2 \rightarrow R_2 - R_1 \\ 2 & 2 & 6-t & t & = \end{array} \right| \left| \begin{array}{cccc} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 2t+4 \\ 0 & 0 & 2-t & t-2 \end{array} \right| \\ &= \left| \begin{array}{ccc|l} 1 & 1 & 3 & R_2 \rightarrow R_2 - 2R_1 \\ 2 & 3 & 2t+4 & = \\ 0 & 2-t & t-2 & \end{array} \right| \left| \begin{array}{ccc} 1 & 1 & 3 \\ 0 & 1 & 2t-2 \\ 0 & 2-t & t-2 \end{array} \right| \\ &= \left| \begin{array}{cc} 1 & 2t-2 \\ 2-t & t-2 \end{array} \right| = (t-2) \left| \begin{array}{cc} 1 & 2t-2 \\ -1 & 1 \end{array} \right| = (t-2)(2t-1). \end{aligned}$$

Hence $\Delta = 0$ if and only if $t = 2$ or $t = \frac{1}{2}$. Consequently the given matrix B is non-singular if and only if $t \neq 2$ and $t \neq \frac{1}{2}$.

11. Let A be a 3×3 matrix with $\det A \neq 0$. Then

(i)

$$\begin{aligned} A \operatorname{adj} A &= (\det A)I_3 \quad (1) \\ (\det A) \det(\operatorname{adj} A) &= \det(\det A \cdot I_3) = (\det A)^3. \end{aligned}$$

Hence, as $\det A \neq 0$, dividing out by $\det A$ in the last equation gives

$$\det(\operatorname{adj} A) = (\det A)^2.$$

(ii) . Also from equation (1)

$$\left(\frac{1}{\det A} A \right) \operatorname{adj} A = I_3,$$

so $\operatorname{adj} A$ is non-singular and

$$(\operatorname{adj} A)^{-1} = \frac{1}{\det A} A.$$

Finally

$$A^{-1} \operatorname{adj}(A^{-1}) = (\det A^{-1})I_3$$

and multiplying both sides of the last equation by A gives

$$\operatorname{adj}(A^{-1}) = A(\det A^{-1})I_3 = \frac{1}{\det A} A.$$

12. Let A be a real 3×3 matrix satisfying $A^t A = I_3$. Then

$$\begin{aligned} \text{(i) } A^t(A - I_3) &= A^t A - A^t = I_3 - A^t \\ &= -(A^t - I_3) = -(A^t - I_3^t) = -(A - I_3)^t. \end{aligned}$$

Taking determinants of both sides then gives

$$\begin{aligned} \det A^t \det(A - I_3) &= \det(-(A - I_3)^t) \\ \det A \det(A - I_3) &= (-1)^3 \det(A - I_3)^t \\ &= -\det(A - I_3) \end{aligned} \quad (1).$$

(ii) Also $\det AA^t = \det I_3$, so

$$\det A^t \det A = 1 = (\det A)^2.$$

Hence $\det A = \pm 1$.

(iii) Suppose that $\det A = 1$. Then equation (1) gives

$$\det(A - I_3) = -\det(A - I_3),$$

so $(1 + 1) \det(A - I_3) = 0$ and hence $\det(A - I_3) = 0$.

13. Suppose that column 1 is a linear combination of the remaining columns:

$$A_{*1} = x_2 A_{*2} + \cdots + x_n A_{*n}.$$

Then

$$\det A = \begin{vmatrix} x_2 a_{12} + \cdots + x_n a_{1n} & a_{12} & \cdots & a_{1n} \\ x_2 a_{22} + \cdots + x_n a_{2n} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_2 a_{n2} + \cdots + x_n a_{nn} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Now $\det A$ is unchanged in value if we perform the operation

$$C_1 \rightarrow C_1 - x_2 C_2 - \cdots - x_n C_n :$$

$$\det A = \begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0.$$

Conversely, suppose that $\det A = 0$. Then the homogeneous system $AX = 0$ has a non-trivial solution $X = [x_1, \dots, x_n]^t$. So

$$x_1 A_{*1} + \dots + x_n A_{*n} = 0.$$

Suppose for example that $x_1 \neq 0$. Then

$$A_{*1} = \left(-\frac{x_2}{x_1}\right) + \dots + \left(-\frac{x_n}{x_1}\right) A_{*n}$$

and the first column of A is a linear combination of the remaining columns.

14. Consider the system

$$\begin{aligned} -2x + 3y - z &= 1 \\ x + 2y - z &= 4 \\ -2x - y + z &= -3 \end{aligned}$$

$$\text{Let } \Delta = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 7 & -3 \\ 1 & 2 & -1 \\ 0 & 3 & -1 \end{vmatrix} = - \begin{vmatrix} 7 & -3 \\ 3 & -1 \end{vmatrix} = -2 \neq 0.$$

Hence the system has a unique solution which can be calculated using Cramer's rule:

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta},$$

where

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix} = -4, \\ \Delta_2 &= \begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix} = -6, \\ \Delta_3 &= \begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix} = -8. \end{aligned}$$

Hence $x = \frac{-4}{-2} = 2$, $y = \frac{-6}{-2} = 3$, $z = \frac{-8}{-2} = 4$.

15. In Remark 4.0.4, take $A = I_n$. Then we deduce

- (a) $\det E_{ij} = -1$;
- (b) $\det E_i(t) = t$;

$$(c) \det E_{ij}(t) = 1.$$

Now suppose that B is a non-singular $n \times n$ matrix. Then we know that B is a product of elementary row matrices:

$$B = E_1 \cdots E_m.$$

Consequently we have to prove that

$$\det E_1 \cdots E_m A = \det E_1 \cdots E_m \det A.$$

We prove this by induction on m .

First the case $m = 1$. We have to prove $\det E_1 A = \det E_1 \det A$ if E_1 is an elementary row matrix. This follows from Remark 4.0.4:

$$(a) \det E_{ij} A = -\det A = \det E_{ij} \det A;$$

$$(b) \det E_i(t) A = t \det A = \det E_i(t) \det A;$$

$$(c) \det E_{ij}(t) A = \det A = \det E_{ij}(t) \det A.$$

Let $m \geq 1$ and assume the proposition holds for products of m elementary row matrices. Then

$$\begin{aligned} \det E_1 \cdots E_m E_{m+1} A &= \det (E_1 \cdots E_m)(E_{m+1} A) \\ &= \det (E_1 \cdots E_m) \det (E_{m+1} A) \\ &= \det (E_1 \cdots E_m) \det E_{m+1} \det A \\ &= \det ((E_1 \cdots E_m) E_{m+1}) \det A \end{aligned}$$

and the induction goes through.

Hence $\det BA = \det B \det A$ if B is non-singular.

If B is singular, problem 26, Chapter 2.7 tells us that BA is also singular. However singular matrices have zero determinant, so

$$\det B = 0 \quad \det BA = 0,$$

so the equation $\det BA = \det B \det A$ holds trivially in this case.

16.

$$\begin{vmatrix} a+b+c & a+b & a & a \\ a+b & a+b+c & a & a \\ a & a & a+b+c & a+b \\ a & a & a+b & a+b+c \end{vmatrix}$$

$$\begin{aligned}
& \begin{array}{l} R_1 \rightarrow R_1 - R_2 \\ R_2 \rightarrow R_2 - R_3 \\ R_3 \rightarrow R_3 - R_4 \\ = \end{array} \left| \begin{array}{cccc} c & -c & 0 & 0 \\ b & b+c & -b-c & -b \\ 0 & 0 & c & -c \\ a & a & a+b & a+b+c \end{array} \right| \\
C_2 \rightarrow C_2 + C_1 & \left| \begin{array}{cccc} c & 0 & 0 & 0 \\ b & 2b+c & -b-c & -b \\ 0 & 0 & c & -c \\ a & 2a & a+b & a+b+c \end{array} \right| = c \left| \begin{array}{ccc} 2b+c & -b-c & -b \\ 0 & c & -c \\ 2a & a+b & a+b+c \end{array} \right| \\
C_3 \rightarrow C_3 + C_2 & c \left| \begin{array}{ccc} 2b+c & -b-c & -2b-c \\ 0 & c & 0 \\ 2a & a+b & 2a+2b+c \end{array} \right| = c^2 \left| \begin{array}{cc} 2b+c & -2b-c \\ 2a & 2a+2b+c \end{array} \right| \\
= c^2(2b+c) & \left| \begin{array}{cc} 1 & -1 \\ 2a & 2a+2b+c \end{array} \right| = c^2(2b+c)(4a+2b+c).
\end{aligned}$$

17. Let $\Delta = \begin{vmatrix} 1+u_1 & u_1 & u_1 & u_1 \\ u_2 & 1+u_2 & u_2 & u_2 \\ u_3 & u_3 & 1+u_3 & u_3 \\ u_4 & u_4 & u_4 & 1+u_4 \end{vmatrix}$. Then using the operation

$$R_1 \rightarrow R_1 + R_2 + R_3 + R_4$$

we have

$$\Delta = \begin{vmatrix} t & t & t & t \\ u_2 & 1+u_2 & u_2 & u_2 \\ u_3 & u_3 & 1+u_3 & u_3 \\ u_4 & u_4 & u_4 & 1+u_4 \end{vmatrix}$$

(where $t = 1 + u_1 + u_2 + u_3 + u_4$)

$$= (1 + u_1 + u_2 + u_3 + u_4) \begin{vmatrix} 1 & 1 & 1 & 1 \\ u_2 & 1+u_2 & u_2 & u_2 \\ u_3 & u_3 & 1+u_3 & u_3 \\ u_4 & u_4 & u_4 & 1+u_4 \end{vmatrix}$$

The last determinant equals

$$\begin{array}{l} C_2 \rightarrow C_2 - C_1 \\ C_3 \rightarrow C_3 - C_1 \\ C_4 \rightarrow C_4 - C_1 \end{array} \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ u_2 & 1 & 0 & 0 \\ u_3 & 0 & 1 & 0 \\ u_4 & 0 & 0 & 1 \end{array} \right| = 1.$$

18. Suppose that $A^t = -A$, that $A \in M_{n \times n}(F)$, where n is odd. Then

$$\begin{aligned}\det A^t &= \det(-A) \\ \det A &= (-1)^n \det A = -\det A.\end{aligned}$$

Hence $(1+1)\det A = 0$ and consequently $\det A = 0$ if $1+1 \neq 0$ in F .

19.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \\ r & r & 1 & 1 \\ r & r & r & 1 \end{vmatrix} = \begin{array}{l} C_4 \rightarrow C_4 - C_3 \\ C_3 \rightarrow C_3 - C_2 \\ C_2 \rightarrow C_2 - C_1 \\ = \end{array} \begin{vmatrix} 1 & 0 & 0 & 0 \\ r & 1-r & 0 & 0 \\ r & 0 & 1-r & 0 \\ r & 0 & 0 & 1-r \end{vmatrix} = (1-r)^3.$$

20.

$$\begin{aligned}& \begin{vmatrix} 1 & a^2 - bc & a^4 \\ 1 & b^2 - ca & b^4 \\ 1 & c^2 - ab & c^4 \end{vmatrix} \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \\ = \end{array} \begin{vmatrix} 1 & a^2 - bc & a^4 \\ 0 & b^2 - ca - a^2 + bc & b^4 - a^4 \\ 0 & c^2 - ab - a^2 + bc & c^4 - a^4 \end{vmatrix} \\ &= \begin{vmatrix} b^2 - ca - a^2 + bc & b^4 - a^4 \\ c^2 - ab - a^2 + bc & c^4 - a^4 \end{vmatrix} \\ &= \begin{vmatrix} (b-a)(b+a) + c(b-a) & (b-a)(b+a)(b^2 + a^2) \\ (c-a)(c+a) + b(c-a) & (c-a)(c+a)(c^2 + a^2) \end{vmatrix} \\ &= \begin{vmatrix} (b-a)(b+a+c) & (b-a)(b+a)(b^2 + a^2) \\ (c-a)(c+a+b) & (c-a)(c+a)(c^2 + a^2) \end{vmatrix} \\ &= (b-a)(c-a) \begin{vmatrix} b+a+c & (b+a)(b^2 + a^2) \\ c+a+b & (c+a)(c^2 + a^2) \end{vmatrix} \\ &= (b-a)(c-a)(a+b+c) \begin{vmatrix} 1 & (b+a)(b^2 + a^2) \\ 1 & (c+a)(c^2 + a^2) \end{vmatrix}.\end{aligned}$$

Finally

$$\begin{aligned}\begin{vmatrix} 1 & (b+a)(b^2 + a^2) \\ 1 & (c+a)(c^2 + a^2) \end{vmatrix} &= (c^3 + ac^2 + ca^2 + a^3) - (b^3 + ab^2 + ba^2 + a^3) \\ &= (c^3 - b^3) + a(c^2 - b^2) + a^2(c - b) \\ &= (c-b)(c^2 + cb + b^2 + a(c+b) + a^2) \\ &= (c-b)(c^2 + cb + b^2 + ac + ab + a^2).\end{aligned}$$