Section 3.6

1. (a) Let $S$ be the set of vectors $[x, y]$ satisfying $x = 2y$. Then $S$ is a vector subspace of $\mathbb{R}^2$. For

(i) $[0, 0] \in S$ as $x = 2y$ holds with $x = 0$ and $y = 0$.

(ii) $S$ is closed under addition. For let $[x_1, y_1]$ and $[x_2, y_2]$ belong to $S$. Then $x_1 = 2y_1$ and $x_2 = 2y_2$. Hence

$$x_1 + x_2 = 2y_1 + 2y_2 = 2(y_1 + y_2)$$

and hence

$$[x_1 + x_2, y_1 + y_2] = [x_1, y_1] + [x_2, y_2]$$

belongs to $S$.

(iii) $S$ is closed under scalar multiplication. For let $[x, y] \in S$ and $t \in \mathbb{R}$. Then $x = 2y$ and hence $tx = 2(ty)$. Consequently

$$[tx, ty] = t[x, y] \in S.$$  

(b) Let $S$ be the set of vectors $[x, y]$ satisfying $x = 2y$ and $2x = y$. Then $S$ is a subspace of $\mathbb{R}^2$. This can be proved in the same way as (a), or alternatively we see that $x = 2y$ and $2x = y$ imply $x = 4y$ and hence $x = 0 = y$. Hence $S = \{(0, 0)\}$, the set consisting of the zero vector. This is always a subspace.

(c) Let $S$ be the set of vectors $[x, y]$ satisfying $x = 2y + 1$. Then $S$ doesn’t contain the zero vector and consequently fails to be a vector subspace.

(d) Let $S$ be the set of vectors $[x, y]$ satisfying $xy = 0$. Then $S$ is not closed under addition of vectors. For example $[1, 0] \in S$ and $[0, 1] \in S$, but $[1, 0] + [0, 1] = [1, 1] \notin S$.

(e) Let $S$ be the set of vectors $[x, y]$ satisfying $x \geq 0$ and $y \geq 0$. Then $S$ is not closed under scalar multiplication. For example $[1, 0] \in S$ and $-1 \in \mathbb{R}$, but $(-1)[1, 0] = [-1, 0] \notin S$.

2. Let $X, Y, Z$ be vectors in $\mathbb{R}^n$. Then by Lemma 3.2.1

$$(X + Y, X + Z, Y + Z) \subseteq (X, Y, Z),$$

as each of $X + Y$, $X + Z$, $Y + Z$ is a linear combination of $X$, $Y$, $Z$. 

Also

\[ X = \frac{1}{2}(X + Y) + \frac{1}{2}(X + Z) - \frac{1}{2}(Y + Z), \]
\[ Y = \frac{1}{2}(X + Y) - \frac{1}{2}(X + Z) + \frac{1}{2}(Y + Z), \]
\[ Z = \frac{-1}{2}(X + Y) + \frac{1}{2}(X + Z) + \frac{1}{2}(Y + Z), \]

so

\[(X, Y, Z) \subseteq (X + Y, X + Z, Y + Z).\]

Hence

\[(X, Y, Z) = (X + Y, X + Z, Y + Z).\]

3. Let \( X_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \) and \( X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \). We have to decide if \( X_1, X_2, X_3 \) are linearly independent, that is if the equation \( xX_1 + yX_2 + zX_3 = 0 \) has only the trivial solution. This equation is equivalent to the following homogeneous system

\[
\begin{align*}
x + 0y + z &= 0 \\
0x + y + z &= 0 \\
x + y + z &= 0 \\
2x + 2y + 3z &= 0.
\end{align*}
\]

We reduce the coefficient matrix to reduced row–echelon form:

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
2 & 2 & 3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

and consequently the system has only the trivial solution \( x = 0, y = 0, z = 0 \). Hence the given vectors are linearly independent.

4. The vectors

\[
X_1 = \begin{bmatrix} \lambda \\ -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ \lambda \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}
\]
are linearly dependent for precisely those values of \( \lambda \) for which the equation 
\[ xX_1 + yX_2 + zX_3 = 0 \]
has a non-trivial solution. This equation is equivalent to the system of homogeneous equations
\[
\begin{align*}
\lambda x - y - z &= 0 \\
-x + \lambda y - z &= 0 \\
-x - y + \lambda z &= 0.
\end{align*}
\]
Now the coefficient determinant of this system is
\[
\begin{vmatrix}
\lambda & -1 & -1 \\
-1 & \lambda & -1 \\
-1 & -1 & \lambda
\end{vmatrix} = (\lambda + 1)^2(\lambda - 2).
\]
So the values of \( \lambda \) which make \( X_1, X_2, X_3 \) linearly independent are those \( \lambda \) satisfying \( \lambda \neq -1 \) and \( \lambda \neq 2 \).

5. Let \( A \) be the following matrix of rationals:
\[
A = \begin{bmatrix}
1 & 1 & 2 & 0 & 1 \\
2 & 2 & 5 & 0 & 3 \\
0 & 0 & 0 & 1 & 3 \\
8 & 11 & 19 & 0 & 11
\end{bmatrix}.
\]
Then \( A \) has reduced row-echelon form
\[
B = \begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 3
\end{bmatrix}.
\]
From \( B \) we read off the following:
(a) The rows of \( B \) form a basis for \( \text{R}(A) \). (Consequently the rows of \( A \) also form a basis for \( \text{R}(A) \).)
(b) The first four columns of \( A \) form a basis for \( \text{C}(A) \).
(c) To find a basis for \( \text{N}(A) \), we solve \( AX = 0 \) and equivalently \( BX = 0 \).
From \( B \) we see that the solution is
\[
\begin{align*}
x_1 &= x_5 \\
x_2 &= 0 \\
x_3 &= -x_5 \\
x_4 &= -3x_5,
\end{align*}
\]
with $x_5$ arbitrary. Then

$$X = \begin{bmatrix} x_5 \\ 0 \\ -x_5 \\ -3x_5 \\ x_5 \end{bmatrix} = x_5 \begin{bmatrix} 1 \\ 0 \\ -1 \\ -3 \\ 1 \end{bmatrix},$$

so $[1, 0, -1, -3, 1]^t$ is a basis for $N(A)$.

6. In Section 1.6, problem 12, we found that the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

has reduced row–echelon form

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From $B$ we read off the following:

(a) The three non–zero rows of $B$ form a basis for $R(A)$.

(b) The first three columns of $A$ form a basis for $C(A)$.

(c) To find a basis for $N(A)$, we solve $AX = 0$ and equivalently $BX = 0$.

From $B$ we see that the solution is

$$x_1 = -x_4 - x_5 = x_4 + x_5$$
$$x_2 = -x_4 - x_5 = x_4 + x_5$$
$$x_3 = -x_4 = x_4,$$

with $x_4$ and $x_5$ arbitrary elements of $\mathbb{Z}_2$. Hence

$$X = \begin{bmatrix} x_4 + x_5 \\ x_4 + x_5 \\ x_4 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $[1, 1, 1, 0]^t$ and $[1, 1, 0, 0, 1]^t$ form a basis for $N(A)$. 33
7. Let $A$ be the following matrix over $\mathbb{Z}_5$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 3 \\ 2 & 1 & 4 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 3 & 2 \end{bmatrix}.$$ 

We find that $A$ has reduced row–echelon form $B$:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 & 4 & 4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{bmatrix}.$$ 

From $B$ we read off the following:

(a) The four rows of $B$ form a basis for $R(A)$. (Consequently the rows of $A$ also form a basis for $R(A)$).

(b) The first four columns of $A$ form a basis for $C(A)$.

(c) To find a basis for $N(A)$, we solve $AX = 0$ and equivalently $BX = 0$.

From $B$ we see that the solution is

$$x_1 = -2x_5 - 4x_6 = 3x_5 + x_6,$$
$$x_2 = -4x_5 - 4x_6 = x_5 + x_6,$$
$$x_3 = 0,$$
$$x_4 = -3x_5 = 2x_5,$$

where $x_5$ and $x_6$ are arbitrary elements of $\mathbb{Z}_5$. Hence

$$X = x_5 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so $[3, 1, 0, 2, 1, 0]^t$ and $[1, 1, 0, 0, 0, 1]^t$ form a basis for $N(A)$.

8. Let $F = \{0, 1, a, b\}$ be a field and let $A$ be the following matrix over $F$:

$$A = \begin{bmatrix} 1 & a & b & a \\ a & b & b & 1 \\ 1 & 1 & 1 & a \end{bmatrix}.$$
In Section 1.6, problem 17, we found that $A$ had reduced row–echelon form

$$B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & b \\
0 & 0 & 1 & 1
\end{bmatrix}. $$

From $B$ we read off the following:

(a) The rows of $B$ form a basis for $R(A)$. (Consequently the rows of $A$ also form a basis for $R(A)$.

(b) The first three columns of $A$ form a basis for $C(A)$.

(c) To find a basis for $N(A)$, we solve $AX = 0$ and equivalently $BX = 0$.

From $B$ we see that the solution is

$$x_1 = 0$$

$$x_2 = -bx_4 = bx_4$$

$$x_3 = -x_4 = x_4,$$

where $x_4$ is an arbitrary element of $F$. Hence

$$X = x_4 \begin{bmatrix} 0 \\ b \\ 1 \\ 1 \end{bmatrix},$$

so $[0, b, 1, 1]^t$ is a basis for $N(A)$.

9. Suppose that $X_1, \ldots, X_m$ form a basis for a subspace $S$. We have to prove that

$$X_1, X_1 + X_2, \ldots, X_1 + \cdots + X_m$$

also form a basis for $S$.

First we prove the independence of the family: Suppose

$$x_1 X_1 + x_2 (X_1 + X_2) + \cdots + x_m (X_1 + \cdots + X_m) = 0.$$  

Then

$$(x_1 + x_2 + \cdots + x_m) X_1 + \cdots + x_m X_m = 0.$$  

Then the linear independence of $X_1, \ldots, X_m$ gives

$$x_1 + x_2 + \cdots + x_m = 0, \ldots, x_m = 0.$$
form which we deduce that \( x_1 = 0, \ldots, x_m = 0 \).

Secondly we have to prove that every vector of \( S \) is expressible as a linear combination of \( X_1, X_1 + X_2, \ldots, X_1 + \cdots + X_m \). Suppose \( X \in S \). Then

\[
X = a_1X_1 + \cdots + a_mX_m.
\]

We have to find \( x_1, \ldots, x_m \) such that

\[
X = x_1X_1 + x_2(X_1 + X_2) + \cdots + x_m(X_1 + \cdots + X_m).
\]

Then

\[
a_1X_1 + \cdots + a_mX_m = (x_1 + x_2 + \cdots + x_m)X_1 + \cdots + x_mX_m.
\]

So if we can solve the system

\[
x_1 + x_2 + \cdots + x_m = a_1, \ldots, x_m = a_m.
\]

we are finished. Clearly these equations have the unique solution

\[
x_1 = a_1 - a_2, \ldots, x_{m-1} = a_m - a_{m-1}, x_m = a_m.
\]

10. Let \( A = \begin{bmatrix} a & b & c \\ 1 & 1 & 1 \end{bmatrix} \). If \([a, b, c]\) is a multiple of \([1, 1, 1]\), (that is, \(a = b = c\)), then \( \text{rank } A = 1 \). For if

\[
[a, b, c] = t[1, 1, 1],
\]

then

\[
\text{R}(A) = \langle [a, b, c], [1, 1, 1] \rangle = \langle t[1, 1, 1], [1, 1, 1] \rangle = \langle [1, 1, 1] \rangle,
\]

so \([1, 1, 1]\) is a basis for \( \text{R}(A) \).

However if \([a, b, c]\) is not a multiple of \([1, 1, 1]\), (that is at least two of \(a, b, c\) are distinct), then the left–to–right test shows that \([a, b, c]\) and \([1, 1, 1]\) are linearly independent and hence form a basis for \( \text{R}(A) \). Consequently \( \text{rank } A = 2 \) in this case.

11. Let \( S \) be a subspace of \( F^n \) with \( \text{dim } S = m \). Also suppose that \( X_1, \ldots, X_m \) are vectors in \( S \) such that \( S = \langle X_1, \ldots, X_m \rangle \). We have to prove that \( X_1, \ldots, X_m \) form a basis for \( S \); in other words, we must prove that \( X_1, \ldots, X_m \) are linearly independent.
However if $X_1, \ldots, X_m$ were linearly dependent, then one of these vectors would be a linear combination of the remaining vectors. Consequently $S$ would be spanned by $m - 1$ vectors. But there exist a family of $m$ linearly independent vectors in $S$. Then by Theorem 3.3.2, we would have the contradiction $m \leq m - 1$.

12. Let $[x, y, z]^t \in S$. Then $x + 2y + 3z = 0$. Hence $x = -2y - 3z$ and

$$
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} = \begin{bmatrix}
  -2y - 3z \\
  y \\
  z
\end{bmatrix} = y \begin{bmatrix}
  -2 \\
  1 \\
  0
\end{bmatrix} + z \begin{bmatrix}
  -3 \\
  0 \\
  1
\end{bmatrix}.
$$

Hence $[−2, 1, 0]^t$ and $[−3, 0, 1]^t$ form a basis for $S$.

Next $−1 + 2(−1) + 3(1) = 0$, so $[−1, −1, 1]^t \in S$.

To find a basis for $S$ which includes $[−1, −1, 1]^t$, we note that $[−2, 1, 0]^t$ is not a multiple of $[−1, −1, 1]^t$. Hence we have found a linearly independent family of two vectors in $S$, a subspace of dimension equal to 2. Consequently these two vectors form a basis for $S$.

13. Without loss of generality, suppose that $X_1 = X_2$. Then we have the non-trivial dependency relation:

$$1X_1 + (−1)X_2 + 0X_3 + \cdots + 0X_m = 0.$$

14. (a) Suppose that $X_{m+1}$ is a linear combination of $X_1, \ldots, X_m$. Then

$$\langle X_1, \ldots, X_m, X_{m+1} \rangle = \langle X_1, \ldots, X_m \rangle$$

and hence

$$\dim \langle X_1, \ldots, X_m, X_{m+1} \rangle = \dim \langle X_1, \ldots, X_m \rangle.$$

(b) Suppose that $X_{m+1}$ is not a linear combination of $X_1, \ldots, X_m$. If not all of $X_1, \ldots, X_m$ are zero, there will be a subfamily $X_{c_1}, \ldots, X_{c_r}$ which is a basis for $\langle X_1, \ldots, X_m \rangle$.

Then as $X_{m+1}$ is not a linear combination of $X_{c_1}, \ldots, X_{c_r}$, it follows that $X_{c_1}, \ldots, X_{c_r}, X_{m+1}$ are linearly independent. Also

$$\langle X_1, \ldots, X_m, X_{m+1} \rangle = \langle X_{c_1}, \ldots, X_{c_r}, X_{m+1} \rangle.$$

Consequently

$$\dim \langle X_1, \ldots, X_m, X_{m+1} \rangle = r + 1 = \dim \langle X_1, \ldots, X_m \rangle + 1.$$
Our result can be rephrased in a form suitable for the second part of the problem:

$$\dim \langle X_1, \ldots, X_m, X_{m+1} \rangle = \dim \langle X_1, \ldots, X_m \rangle$$

if and only if $X_{m+1}$ is a linear combination of $X_1, \ldots, X_m$.

If $X = [x_1, \ldots, x_n]^t$, then $AX = B$ is equivalent to

$$B = x_1A_{s1} + \cdots + x_nA_{sn}.$$ 

So $AX = B$ is soluble for $X$ if and only if $B$ is a linear combination of the columns of $A$, that is $B \in C(A)$. However by the first part of this question, $B \in C(A)$ if and only if $\dim C([A|B]) = \dim C(A)$, that is, rank $[A|B] = \text{rank } A$.

15. Let $a_1, \ldots, a_n$ be elements of $F$, not all zero. Let $S$ denote the set of vectors $[x_1, \ldots, x_n]^t$, where $x_1, \ldots, x_n$ satisfy

$$a_1x_1 + \cdots + a_nx_n = 0.$$ 

Then $S = N(A)$, where $A$ is the row matrix $[a_1, \ldots, a_n]$. Now rank $A = 1$ as $A \neq 0$. So by the “rank + nullity” theorem, noting that the number of columns of $A$ equals $n$, we have

$$\dim N(A) = \text{nullity } (A) = n - \text{rank } A = n - 1.$$ 

16. (a) (Proof of Lemma 3.2.1) Suppose that each of $X_1, \ldots, X_r$ is a linear combination of $Y_1, \ldots, Y_s$. Then

$$X_i = \sum_{j=1}^s a_{ij}Y_j, \quad (1 \leq i \leq r).$$

Now let $X = \sum_{i=1}^r x_iX_i$ be a linear combination of $X_1, \ldots, X_r$. Then

\[
X = x_1(a_{11}Y_1 + \cdots + a_{1s}Y_s) \\
+ \cdots \\
+ x_r(a_{r1}Y_1 + \cdots + a_{rs}Y_s) \\
= y_1Y_1 + \cdots + y_sY_s,
\]

where $y_j = a_{1j}x_1 + \cdots + a_{rj}x_r$. Hence $X$ is a linear combination of $Y_1, \ldots, Y_s$.

Another way of stating Lemma 3.2.1 is

$$\langle X_1, \ldots, X_r \rangle \subseteq \langle Y_1, \ldots, Y_s \rangle, \quad (1)$$

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if each of \( X_1, \ldots, X_r \) is a linear combination of \( Y_1, \ldots, Y_s \).

(b) (Proof of Theorem 3.2.1) Suppose that each of \( X_1, \ldots, X_r \) is a linear combination of \( Y_1, \ldots, Y_s \) and that each of \( Y_1, \ldots, Y_s \) is a linear combination of \( X_1, \ldots, X_r \). Then by (a) equation (1) above

\[
\langle X_1, \ldots, X_r \rangle \subseteq \langle Y_1, \ldots, Y_s \rangle
\]

and

\[
\langle Y_1, \ldots, Y_s \rangle \subseteq \langle X_1, \ldots, X_r \rangle.
\]

Hence

\[
\langle X_1, \ldots, X_r \rangle = \langle Y_1, \ldots, Y_s \rangle.
\]

(c) (Proof of Corollary 3.2.1) Suppose that each of \( Z_1, \ldots, Z_t \) is a linear combination of \( X_1, \ldots, X_r \). Then each of \( X_1, \ldots, X_r, Z_1, \ldots, Z_t \) is a linear combination of \( X_1, \ldots, X_r \).

Also each of \( X_1, \ldots, X_r \) is a linear combination of \( X_1, \ldots, X_r, Z_1, \ldots, Z_t \), so by Theorem 3.2.1

\[
\langle X_1, \ldots, X_r, Z_1, \ldots, Z_t \rangle = \langle X_1, \ldots, X_r \rangle.
\]

(d) (Proof of Theorem 3.3.2) Let \( Y_1, \ldots, Y_s \) be vectors in \( \langle X_1, \ldots, X_r \rangle \) and assume that \( s > r \). We have to prove that \( Y_1, \ldots, Y_s \) are linearly dependent. So we consider the equation

\[
x_1 Y_1 + \cdots + x_s Y_s = 0.
\]

Now \( Y_i = \sum_{j=1}^r a_{ij} X_j \), for \( 1 \leq i \leq s \). Hence

\[
x_1 Y_1 + \cdots + x_s Y_s = x_1 (a_{11} X_1 + \cdots + a_{1r} X_r) + \cdots + x_r (a_{s1} X_1 + \cdots + a_{sr} X_r).
\]

\[
= y_1 X_1 + \cdots + y_r X_r, \quad (1)
\]

where \( y_j = a_{1j} x_1 + \cdots + a_{sj} x_s \). However the homogeneous system

\[
y_1 = 0, \; \cdots, \; y_r = 0
\]

has a non–trivial solution \( x_1, \ldots, x_s \), as \( s > r \) and from (1), this results in a non–trivial solution of the equation

\[
x_1 Y_1 + \cdots + x_s Y_s = 0.
\]

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Hence \( Y_1, \ldots, Y_s \) are linearly dependent.

17. Let \( R \) and \( S \) be subspaces of \( F^n \), with \( R \subseteq S \). We first prove

\[
\dim R \leq \dim S.
\]

Let \( X_1, \ldots, X_r \) be a basis for \( R \). Now by Theorem 3.5.2, because \( X_1, \ldots, X_r \) form a linearly independent family lying in \( S \), this family can be extended to a basis \( X_1, \ldots, X_r, \ldots, X_s \) for \( S \). Then

\[
\dim S = s \geq r = \dim R.
\]

Next suppose that \( \dim R = \dim S \). Let \( X_1, \ldots, X_r \) be a basis for \( R \). Then because \( X_1, \ldots, X_r \) form a linearly independent family in \( S \) and \( S \) is a subspace whose dimension is \( r \), it follows from Theorem 3.4.3 that \( X_1, \ldots, X_r \) form a basis for \( S \). Then

\[
S = \langle X_1, \ldots, X_r \rangle = R.
\]

18. Suppose that \( R \) and \( S \) are subspaces of \( F^n \) with the property that \( R \cup S \) is also a subspace of \( F^n \). We have to prove that \( R \subseteq S \) or \( S \subseteq R \). We argue by contradiction: Suppose that \( R \not\subseteq S \) and \( S \not\subseteq R \). Then there exist vectors \( u \) and \( v \) such that

\[
u \in R \text{ and } u \notin S, \quad v \in S \text{ and } v \notin R.
\]

Consider the vector \( u + v \). As we are assuming \( R \cup S \) is a subspace, \( R \cup S \) is closed under addition. Hence \( u + v \in R \cup S \) and so \( u + v \in R \) or \( u + v \in S \). However if \( u + v \in R \), then \( v = (u + v) - u \in R \), which is a contradiction; similarly if \( u + v \in S \).

Hence we have derived a contradiction on the assumption that \( R \not\subseteq S \) and \( S \not\subseteq R \). Consequently at least one of these must be false. In other words \( R \subseteq S \) or \( S \subseteq R \).

19. Let \( X_1, \ldots, X_r \) be a basis for \( S \).

(i) First let

\[
\begin{align*}
Y_1 &= a_{11}X_1 + \cdots + a_{1r}X_r \\
\vdots \\
Y_r &= a_{r1}X_1 + \cdots + a_{rr}X_r,
\end{align*}
\]
where $A = [a_{ij}]$ is non-singular. Then the above system of equations can be solved for $X_1, \ldots, X_r$ in terms of $Y_1, \ldots, Y_r$. Consequently by Theorem 3.2.1

$$ (Y_1, \ldots, Y_r) = (X_1, \ldots, X_r) = S. $$

It follows from problem 11 that $Y_1, \ldots, Y_r$ is a basis for $S$.

(ii) We show that all bases for $S$ are given by equations 2. So suppose that $Y_1, \ldots, Y_r$ forms a basis for $S$. Then because $X_1, \ldots, X_r$ form a basis for $S$, we can express $Y_1, \ldots, Y_r$ in terms of $X_1, \ldots, X_r$ as in 2, for some matrix $A = [a_{ij}]$. We show $A$ is non-singular by demonstrating that the linear independence of $Y_1, \ldots, Y_r$ implies that the rows of $A$ are linearly independent.

So assume

$$ x_1[a_{11}, \ldots, a_{1r}] + \cdots + x_r[a_{r1}, \ldots, a_{rr}] = [0, \ldots, 0]. $$

Then on equating components, we have

$$ a_{11}x_1 + \cdots + a_{1r}x_r = 0 $$

$$ \vdots $$

$$ a_{1r}x_1 + \cdots + a_{rr}x_r = 0. $$

Hence

$$ x_1Y_1 + \cdots + x_rY_r = x_1(a_{11}X_1 + \cdots + a_{1r}X_r) + \cdots + x_r(a_{r1}X_1 + \cdots + a_{rr}X_r) $$

$$ = (a_{11}x_1 + \cdots + a_{1r}x_r)X_1 + \cdots + (a_{r1}x_1 + \cdots + a_{rr}x_r)X_r $$

$$ = 0X_1 + \cdots + 0X_r = 0. $$

Then the linear independence of $Y_1, \ldots, Y_r$ implies $x_1 = 0, \ldots, x_r = 0$.

(We mention that the last argument is reversible and provides an alternative proof of part (i).)