SECTION 1.6

2. (i) $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_1 \to \frac{1}{2}R_1 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

(ii) $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$;

(iii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix}$

$\begin{matrix} R_1 \to R_1 + R_3 \\ R_3 \to -R_3 \\ R_2 \leftrightarrow R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_3 \\ R_3 \to -R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

(iv) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 + 2R_1 \\ R_1 \to \frac{1}{2}R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

3. (a) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & -1 & 8 \\ 1 & -1 & -1 & -8 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & -2 & -2 & -10 \end{bmatrix}$

$\begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 2R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 4 & -2 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & -8 & -2 \end{bmatrix} R_3 \to \frac{-1}{8}R_3 \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix}$

$\begin{matrix} R_1 \to R_1 - 4R_3 \\ R_2 \to R_2 + 3R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & \frac{19}{4} \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix}$.

The augmented matrix has been converted to reduced row–echelon form and we read off the unique solution $x = -3,\ y = \frac{19}{4},\ z = \frac{1}{4}$.

(b) $\begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 3 & -1 & 7 & 4 & 1 \\ -5 & 3 & -15 & -6 & 9 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 5R_1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & -4 & 10 & -2 & -29 \\ 0 & 8 & -20 & 4 & 59 \end{bmatrix}$

$R_3 \to R_3 + 2R_2 \begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & -4 & 10 & -2 & -29 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

From the last matrix we see that the original system is inconsistent.

(c) $\begin{bmatrix} 3 & -1 & 7 & 0 \\ 2 & -1 & 4 & \frac{1}{2} \\ 1 & -1 & 1 & 1 \\ 6 & -4 & 10 & 3 \end{bmatrix} R_1 \leftrightarrow R_3 \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 4 & \frac{1}{2} \\ 3 & -1 & 7 & 0 \\ 6 & -4 & 10 & 3 \end{bmatrix}$

$$\begin{array}{c} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{array} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & \frac{-3}{2} \\ 0 & 2 & 4 & -3 \\ 0 & 2 & 4 & -3 \end{bmatrix} \begin{array}{c} R_1 \to R_1 + R_2 \\ R_4 \to R_4 - R_3 \\ R_3 \to R_3 - 2R_2 \end{array} \begin{bmatrix} 1 & 0 & 3 & \frac{-1}{2} \\ 0 & 1 & 2 & \frac{-3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The augmented matrix has been converted to reduced row–echelon form and we read off the complete solution $x = -\frac{1}{2} - 3z,\ y = -\frac{3}{2} - 2z$, with z arbitrary.

4. $\begin{bmatrix} 2 & -1 & 3 & a \\ 3 & 1 & -5 & b \\ -5 & -5 & 21 & c \end{bmatrix} R_2 \to R_2 - R_1 \begin{bmatrix} 2 & -1 & 3 & a \\ 1 & 2 & -8 & b-a \\ -5 & -5 & 21 & c \end{bmatrix}$

$$R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 2 & -8 & b-a \\ 2 & -1 & 3 & a \\ -5 & -5 & 21 & c \end{bmatrix} \begin{array}{c} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 5R_1 \end{array} \begin{bmatrix} 1 & 2 & -8 & b-a \\ 0 & -5 & 19 & -2b+3a \\ 0 & 5 & -19 & 5b-5a+c \end{bmatrix}$$

$$\begin{array}{c} R_3 \to R_3 + R_2 \\ R_2 \to \frac{-1}{5} R_2 \end{array} \begin{bmatrix} 1 & 2 & -8 & b-a \\ 0 & 1 & \frac{-19}{5} & \frac{2b-3a}{5} \\ 0 & 0 & 0 & 3b-2a+c \end{bmatrix}$$

$$R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & \frac{-2}{5} & \frac{(b+a)}{5} \\ 0 & 1 & \frac{-19}{5} & \frac{2b-3a}{5} \\ 0 & 0 & 0 & 3b-2a+c \end{bmatrix}.$$

From the last matrix we see that the original system is inconsistent if $3b - 2a + c \neq 0$. If $3b - 2a + c = 0$, the system is consistent and the solution is

$$x = \frac{(b+a)}{5} + \frac{2}{5}z,\ y = \frac{(2b-3a)}{5} + \frac{19}{5}z,$$

where z is arbitrary.

5. $\begin{bmatrix} 1 & 1 & 1 \\ t & 1 & t \\ 1+t & 2 & 3 \end{bmatrix} \begin{array}{c} R_2 \to R_2 - tR_1 \\ R_3 \to R_3 - (1+t)R_1 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1-t & 0 \\ 0 & 1-t & 2-t \end{bmatrix}$

$$R_3 \to R_3 - R_2 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1-t & 0 \\ 0 & 0 & 2-t \end{bmatrix} = B.$$

Case 1. $t \neq 2$. No solution.

Case 2. $\quad t=2.\ B=\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

We read off the unique solution $x=1,\ y=0$.

6. Method 1.

$$\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} \begin{matrix} R_1 \rightarrow R_1 - R_4 \\ R_2 \rightarrow R_2 - R_4 \\ R_3 \rightarrow R_3 - R_4 \end{matrix} \begin{bmatrix} -4 & 0 & 0 & 4 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & -4 & 4 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & -3 \end{bmatrix} R_4 \rightarrow R_4 - R_3 - R_2 - R_1 \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the given homogeneous system has complete solution

$$x_1 = x_4,\ x_2 = x_4,\ x_3 = x_4,$$

with x_4 arbitrary.

Method 2. Write the system as

$$\begin{aligned} x_1+x_2+x_3+x_4 &= 4x_1 \\ x_1+x_2+x_3+x_4 &= 4x_2 \\ x_1+x_2+x_3+x_4 &= 4x_3 \\ x_1+x_2+x_3+x_4 &= 4x_4. \end{aligned}$$

Then it is immediate that any solution must satisfy $x_1 = x_2 = x_3 = x_4$. Conversely, if x_1, x_2, x_3, x_4 satisfy $x_1 = x_2 = x_3 = x_4$, we get a solution.

7.

$$\begin{bmatrix} \lambda-3 & 1 \\ 1 & \lambda-3 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & \lambda-3 \\ \lambda-3 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - (\lambda-3)R_1 \begin{bmatrix} 1 & \lambda-3 \\ 0 & -\lambda^2+6\lambda-8 \end{bmatrix} = B.$$

Case 1: $-\lambda^2+6\lambda-8 \neq 0$. That is $-(\lambda-2)(\lambda-4) \neq 0$ or $\lambda \neq 2, 4$. Here B is row equivalent to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

$$R_2 \rightarrow \tfrac{1}{-\lambda^2+6\lambda-8} R_2 \begin{bmatrix} 1 & \lambda-3 \\ 0 & 1 \end{bmatrix} R_1 \rightarrow R_1 - (\lambda-3)R_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence we get the trivial solution $x=0,\ y=0$.

Case 2: $\lambda = 2$. Then $B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ and the solution is $x = y$, with y arbitrary.

Case 3: $\lambda = 4$. Then $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and the solution is $x = -y$, with y arbitrary.

8.

$$\begin{aligned}
\begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix} R_1 &\to \frac{1}{3}R_1 \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 5 & -1 & 1 & -1 \end{bmatrix} \\
R_2 &\to R_2 - 5R_1 \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} \end{bmatrix} \\
R_2 &\to \frac{-3}{8}R_2 \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{4} & 1 \end{bmatrix} \\
R_1 &\to R_1 - \frac{1}{3}R_2 \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{4} & 1 \end{bmatrix}.
\end{aligned}$$

Hence the solution of the associated homogeneous system is

$$x_1 = -\frac{1}{4}x_3, \ x_2 = -\frac{1}{4}x_3 - x_4,$$

with x_3 and x_4 arbitrary.

9.

$$A = \begin{bmatrix} 1-n & 1 & \cdots & 1 \\ 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1-n \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_n \\ R_2 \to R_2 - R_n \\ \vdots \\ R_{n-1} \to R_{n-1} - R_n \end{matrix} \begin{bmatrix} -n & 0 & \cdots & n \\ 0 & -n & \cdots & n \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1-n \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \cdots & -1 \\ 0 & 1 & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1-n \end{bmatrix} R_n \to R_n - R_{n-1} \cdots - R_1 \begin{bmatrix} 1 & 0 & \cdots & -1 \\ 0 & 1 & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The last matrix is in reduced row–echelon form.

Consequently the homogeneous system with coefficient matrix A has the solution

$$x_1 = x_n, \ x_2 = x_n, \ldots, x_{n-1} = x_n,$$

with x_n arbitrary.

Alternatively, writing the system in the form

$$\begin{array}{rcl} x_1 + \cdots + x_n &=& nx_1 \\ x_1 + \cdots + x_n &=& nx_2 \\ &\vdots& \\ x_1 + \cdots + x_n &=& nx_n \end{array}$$

shows that any solution must satisfy $nx_1 = nx_2 = \cdots = nx_n$, so $x_1 = x_2 = \cdots = x_n$. Conversely if $x_1 = x_n, \ldots, x_{n-1} = x_n$, we see that $x_1, \ldots, x_n$ is a solution.

10. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and assume that $ad - bc \neq 0$.

Case 1: $a \neq 0$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} R_1 \to \tfrac{1}{a}R_1 \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} R_2 \to R_2 - cR_1 \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix}$$

$$R_2 \to \tfrac{a}{ad-bc}R_2 \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix} R_1 \to R_1 - \tfrac{b}{a}R_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Case 2: $a = 0$. Then $bc \neq 0$ and hence $c \neq 0$.

$$A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \to \begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So in both cases, A has reduced row–echelon form equal to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

11. We simplify the augmented matrix of the system using row operations:

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \end{array} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$$

$$\begin{array}{r} R_3 \to R_3 - R_2 \\ R_2 \to \frac{-1}{7}R_2 \\ R_1 \to R_1 - 2R_2 \end{array} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix} R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}.$$

Denote the last matrix by B.

Case 1: $a^2 - 16 \neq 0$. i.e. $a \neq \pm 4$. Then

$$\begin{array}{c} R_3 \to \frac{1}{a^2-16} R_3 \\ R_1 \to R_1 - R_3 \\ R_2 \to R_2 + 2R_3 \end{array} \left[\begin{array}{cccc} 1 & 0 & 0 & \frac{8a+25}{7(a+4)} \\ 0 & 1 & 0 & \frac{10a+54}{7(a+4)} \\ 0 & 0 & 1 & \frac{1}{a+4} \end{array}\right]$$

and we get the unique solution

$$x = \frac{8a+25}{7(a+4)},\ y = \frac{10a+54}{7(a+4)},\ z = \frac{1}{a+4}.$$

Case 2: $a = -4$. Then $B = \left[\begin{array}{cccc} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & 0 & -8 \end{array}\right]$, so our system is inconsistent.

Case 3: $a = 4$. Then $B = \left[\begin{array}{cccc} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$. We read off that the system is consistent, with complete solution $x = \frac{8}{7} - z,\ y = \frac{10}{7} + 2z$, where z is arbitrary.

12. We reduce the augmented array of the system to reduced row–echelon form:

$$\left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad R_3 \to R_3 + R_1 \quad \left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 \to R_3 + R_2 \quad \left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{c} R_1 \to R_1 + R_4 \\ R_3 \leftrightarrow R_4 \end{array} \quad \left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The last matrix is in reduced row–echelon form and we read off the solution of the corresponding homogeneous system:

$$\begin{aligned} x_1 &= -x_4 - x_5 = x_4 + x_5 \\ x_2 &= -x_4 - x_5 = x_4 + x_5 \\ x_3 &= -x_4 = x_4, \end{aligned}$$

where x_4 and x_5 are arbitrary elements of $\mathbb{Z}_2$. Hence there are four solutions:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}.$$

13. (a) We reduce the augmented matrix to reduced row–echelon form:

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 1 & 4 & 1 \\ 3 & 1 & 2 & 0 \end{bmatrix} \quad R_1 \to 3R_1 \quad \begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 1 & 4 & 1 \\ 3 & 1 & 2 & 0 \end{bmatrix}$$

$$\begin{array}{c} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 2R_1 \end{array} \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 4 & 3 & 3 \\ 0 & 2 & 0 & 4 \end{bmatrix} \quad R_2 \to 4R_2 \quad \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$$\begin{array}{c} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 + 3R_2 \end{array} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{c} R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 + 3R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Consequently the system has the unique solution $x = 1$, $y = 2$, $z = 0$.

(b) Again we reduce the augmented matrix to reduced row–echelon form:

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 1 & 4 & 1 \\ 1 & 1 & 0 & 3 \end{bmatrix} \quad R_1 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 1 & 0 & 3 \\ 4 & 1 & 4 & 1 \\ 2 & 1 & 3 & 4 \end{bmatrix}$$

$$\begin{array}{c} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 3R_1 \end{array} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 4 & 4 \\ 0 & 4 & 3 & 3 \end{bmatrix} \quad R_2 \to 3R_2 \quad \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 4 & 3 & 3 \end{bmatrix}$$

$$\begin{array}{c} R_1 \to R_1 + 4R_2 \\ R_3 \to R_3 + R_2 \end{array} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We read off the complete solution

$$\begin{aligned} x &= 1 - 3z = 1 + 2z \\ y &= 2 - 2z = 2 + 3z, \end{aligned}$$

where z is an arbitrary element of $\mathbb{Z}_5$.

14. Suppose that $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ are solutions of the system of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad 1 \le i \le m.$$

Then

$$\sum_{j=1}^{n} a_{ij} \alpha_j = b_i \quad \text{and} \quad \sum_{j=1}^{n} a_{ij} \beta_j = b_i$$

for $1 \le i \le m$.

Let $\gamma_i = (1-t)\alpha_i + t\beta_i$ for $1 \le i \le m$. Then $(\gamma_1, \ldots, \gamma_n)$ is a solution of the given system. For

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} \gamma_j &= \sum_{j=1}^{n} a_{ij} \{(1-t)\alpha_j + t\beta_j\} \\ &= \sum_{j=1}^{n} a_{ij}(1-t)\alpha_j + \sum_{j=1}^{n} a_{ij} t\beta_j \\ &= (1-t)b_i + tb_i \\ &= b_i. \end{aligned}$$

15. Suppose that $(\alpha_1, \ldots, \alpha_n)$ is a solution of the system of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad 1 \le i \le m. \tag{1}$$

Then the system can be rewritten as

$$\sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} a_{ij} \alpha_j, \quad 1 \le i \le m,$$

or equivalently

$$\sum_{j=1}^{n} a_{ij}(x_j - \alpha_j) = 0, \quad 1 \le i \le m.$$

So we have

$$\sum_{j=1}^{n} a_{ij} y_j = 0, \quad 1 \le i \le m.$$

where $x_j - \alpha_j = y_j$. Hence $x_j = \alpha_j + y_j$, $1 \le j \le n$, where $(y_1, \ldots, y_n)$ is a solution of the associated homogeneous system. Conversely if $(y_1, \ldots, y_n)$

is a solution of the associated homogeneous system and $x_j = \alpha_j + y_j,\ 1 \le j \le n$, then reversing the argument shows that $(x_1, \ldots, x_n)$ is a solution of the system 2 .

16. We simplify the augmented matrix using row operations, working towards row–echelon form:

$$\left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ a & 1 & 1 & 1 & b \\ 3 & 2 & 0 & a & 1+a \end{array}\right] \begin{array}{c} R_2 \to R_2 - aR_1 \\ R_3 \to R_3 - 3R_1 \end{array} \left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ 0 & 1-a & 1+a & 1-a & b-a \\ 0 & -1 & 3 & a-3 & a-2 \end{array}\right]$$

$$\begin{array}{c} R_2 \leftrightarrow R_3 \\ R_2 \to -R_2 \end{array} \left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 3-a & 2-a \\ 0 & 1-a & 1+a & 1-a & b-a \end{array}\right]$$

$$R_3 \to R_3 + (a-1)R_2 \left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 3-a & 2-a \\ 0 & 0 & 4-2a & (1-a)(a-2) & -a^2+2a+b-2 \end{array}\right] = B.$$

Case 1: $a \neq 2$. Then $4 - 2a \neq 0$ and

$$B \to \left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 3-a & 2-a \\ 0 & 0 & 1 & \frac{a-1}{2} & \frac{-a^2+2a+b-2}{4-2a} \end{array}\right].$$

Hence we can solve for x, y and z in terms of the arbitrary variable w.

Case 2: $a = 2$. Then

$$B = \left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & b-2 \end{array}\right].$$

Hence there is no solution if $b \neq 2$. However if $b = 2$, then

$$B = \left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and we get the solution $x = 1 - 2z,\ y = 3z - w$, where w is arbitrary.

17. (a) We first prove that $1 + 1 + 1 + 1 = 0$. Observe that the elements

$$1+0, \quad 1+1, \quad 1+a, \quad 1+b$$

are distinct elements of F by virtue of the *cancellation law for addition.* For this law states that $1+x = 1+y \Rightarrow x = y$ and hence $x \neq y \Rightarrow 1+x \neq 1+y$.

Hence the above four elements are just the elements 0, 1, a, b in some order. Consequently

$$\begin{aligned} (1+0)+(1+1)+(1+a)+(1+b) &= 0+1+a+b \\ (1+1+1+1)+(0+1+a+b) &= 0+(0+1+a+b), \end{aligned}$$

so $1+1+1+1 = 0$ after cancellation.

Now $1+1+1+1 = (1+1)(1+1)$, so we have $x^2 = 0$, where $x = 1+1$. Hence $x = 0$. Then $a+a = a(1+1) = a \cdot 0 = 0$.

Next $a+b = 1$. For $a+b$ must be one of 0, 1, a, b. Clearly we can't have $a+b = a$ or b; also if $a+b = 0$, then $a+b = a+a$ and hence $b = a$; hence $a+b = 1$. Then

$$a+1 = a+(a+b) = (a+a)+b = 0+b = b.$$

Similarly $b+1 = a$. Consequently the addition table for F is

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

.

We now find the multiplication table. First, ab must be one of 1, a, b; however we can't have $ab = a$ or b, so this leaves $ab = 1$.

Next $a^2 = b$. For a^2 must be one of 1, a, b; however $a^2 = a \Rightarrow a = 0$ or $a = 1$; also

$$a^2 = 1 \Rightarrow a^2 - 1 = 0 \Rightarrow (a-1)(a+1) = 0 \Rightarrow (a-1)^2 = 0 \Rightarrow a = 1;$$

hence $a^2 = b$. Similarly $b^2 = a$. Consequently the multiplication table for F is

×	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

.

(b) We use the addition and multiplication tables for F:

$$A = \begin{bmatrix} 1 & a & b & a \\ a & b & b & 1 \\ 1 & 1 & 1 & a \end{bmatrix} \quad \begin{array}{c} R_2 \to R_2 + aR_1 \\ R_3 \to R_3 + R_1 \end{array} \quad \begin{bmatrix} 1 & a & b & a \\ 0 & 0 & a & a \\ 0 & b & a & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & a & b & a \\ 0 & b & a & 0 \\ 0 & 0 & a & a \end{bmatrix} \begin{matrix} R_2 \to aR_2 \\ R_3 \to bR_3 \end{matrix} \begin{bmatrix} 1 & a & b & a \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_1 + aR_2 \quad \begin{bmatrix} 1 & 0 & a & a \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 + aR_3 \\ R_2 \to R_2 + bR_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The last matrix is in reduced row–echelon form.

SECTION 1.6

2. (i) $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_1 \to \frac{1}{2}R_1 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

(ii) $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$;

(iii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix}$

$\begin{matrix} R_1 \to R_1 + R_3 \\ R_3 \to -R_3 \\ R_2 \leftrightarrow R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_3 \\ R_3 \to -R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

(iv) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 + 2R_1 \\ R_1 \to \frac{1}{2}R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

3. (a) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & -1 & 8 \\ 1 & -1 & -1 & -8 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & -2 & -2 & -10 \end{bmatrix}$

$\begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 2R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 4 & -2 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & -8 & -2 \end{bmatrix} R_3 \to \frac{-1}{8}R_3 \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix}$

$\begin{matrix} R_1 \to R_1 - 4R_3 \\ R_2 \to R_2 + 3R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & \frac{19}{4} \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix}$.

The augmented matrix has been converted to reduced row–echelon form and we read off the unique solution $x = -3,\ y = \frac{19}{4},\ z = \frac{1}{4}$.

(b) $\begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 3 & -1 & 7 & 4 & 1 \\ -5 & 3 & -15 & -6 & 9 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 5R_1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & -4 & 10 & -2 & -29 \\ 0 & 8 & -20 & 4 & 59 \end{bmatrix}$

$R_3 \to R_3 + 2R_2 \begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & -4 & 10 & -2 & -29 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

From the last matrix we see that the original system is inconsistent.

(c) $\begin{bmatrix} 3 & -1 & 7 & 0 \\ 2 & -1 & 4 & \frac{1}{2} \\ 1 & -1 & 1 & 1 \\ 6 & -4 & 10 & 3 \end{bmatrix} R_1 \leftrightarrow R_3 \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 4 & \frac{1}{2} \\ 3 & -1 & 7 & 0 \\ 6 & -4 & 10 & 3 \end{bmatrix}$

$$\begin{array}{c} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{array} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & \frac{-3}{2} \\ 0 & 2 & 4 & -3 \\ 0 & 2 & 4 & -3 \end{bmatrix} \begin{array}{c} R_1 \to R_1 + R_2 \\ R_4 \to R_4 - R_3 \\ R_3 \to R_3 - 2R_2 \end{array} \begin{bmatrix} 1 & 0 & 3 & \frac{-1}{2} \\ 0 & 1 & 2 & \frac{-3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The augmented matrix has been converted to reduced row–echelon form and we read off the complete solution $x = -\frac{1}{2} - 3z,\ y = -\frac{3}{2} - 2z$, with z arbitrary.

4. $\begin{bmatrix} 2 & -1 & 3 & a \\ 3 & 1 & -5 & b \\ -5 & -5 & 21 & c \end{bmatrix} R_2 \to R_2 - R_1 \begin{bmatrix} 2 & -1 & 3 & a \\ 1 & 2 & -8 & b-a \\ -5 & -5 & 21 & c \end{bmatrix}$

$$R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 2 & -8 & b-a \\ 2 & -1 & 3 & a \\ -5 & -5 & 21 & c \end{bmatrix} \begin{array}{c} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 5R_1 \end{array} \begin{bmatrix} 1 & 2 & -8 & b-a \\ 0 & -5 & 19 & -2b+3a \\ 0 & 5 & -19 & 5b-5a+c \end{bmatrix}$$

$$\begin{array}{c} R_3 \to R_3 + R_2 \\ R_2 \to \frac{-1}{5} R_2 \end{array} \begin{bmatrix} 1 & 2 & -8 & b-a \\ 0 & 1 & \frac{-19}{5} & \frac{2b-3a}{5} \\ 0 & 0 & 0 & 3b-2a+c \end{bmatrix}$$

$$R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & \frac{-2}{5} & \frac{(b+a)}{5} \\ 0 & 1 & \frac{-19}{5} & \frac{2b-3a}{5} \\ 0 & 0 & 0 & 3b-2a+c \end{bmatrix}.$$

From the last matrix we see that the original system is inconsistent if $3b - 2a + c \neq 0$. If $3b - 2a + c = 0$, the system is consistent and the solution is

$$x = \frac{(b+a)}{5} + \frac{2}{5}z,\ y = \frac{(2b-3a)}{5} + \frac{19}{5}z,$$

where z is arbitrary.

5. $\begin{bmatrix} 1 & 1 & 1 \\ t & 1 & t \\ 1+t & 2 & 3 \end{bmatrix} \begin{array}{c} R_2 \to R_2 - tR_1 \\ R_3 \to R_3 - (1+t)R_1 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1-t & 0 \\ 0 & 1-t & 2-t \end{bmatrix}$

$$R_3 \to R_3 - R_2 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1-t & 0 \\ 0 & 0 & 2-t \end{bmatrix} = B.$$

Case 1. $t \neq 2$. No solution.

Case 2. $t = 2$. $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

We read off the unique solution $x = 1,\ y = 0$.

6. Method 1.

$$\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} \begin{array}{l} R_1 \rightarrow R_1 - R_4 \\ R_2 \rightarrow R_2 - R_4 \\ R_3 \rightarrow R_3 - R_4 \end{array} \begin{bmatrix} -4 & 0 & 0 & 4 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & -4 & 4 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & -3 \end{bmatrix} R_4 \rightarrow R_4 - R_3 - R_2 - R_1 \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the given homogeneous system has complete solution

$$x_1 = x_4,\ x_2 = x_4,\ x_3 = x_4,$$

with x_4 arbitrary.

Method 2. Write the system as

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 4x_1 \\ x_1 + x_2 + x_3 + x_4 &= 4x_2 \\ x_1 + x_2 + x_3 + x_4 &= 4x_3 \\ x_1 + x_2 + x_3 + x_4 &= 4x_4. \end{aligned}$$

Then it is immediate that any solution must satisfy $x_1 = x_2 = x_3 = x_4$. Conversely, if x_1, x_2, x_3, x_4 satisfy $x_1 = x_2 = x_3 = x_4$, we get a solution.

7.

$$\begin{bmatrix} \lambda - 3 & 1 \\ 1 & \lambda - 3 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & \lambda - 3 \\ \lambda - 3 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - (\lambda - 3)R_1 \begin{bmatrix} 1 & \lambda - 3 \\ 0 & -\lambda^2 + 6\lambda - 8 \end{bmatrix} = B.$$

Case 1: $-\lambda^2 + 6\lambda - 8 \neq 0$. That is $-(\lambda - 2)(\lambda - 4) \neq 0$ or $\lambda \neq 2, 4$. Here B is row equivalent to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

$$R_2 \rightarrow \frac{1}{-\lambda^2+6\lambda-8} R_2 \begin{bmatrix} 1 & \lambda - 3 \\ 0 & 1 \end{bmatrix} R_1 \rightarrow R_1 - (\lambda - 3)R_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence we get the trivial solution $x = 0,\ y = 0$.

Case 2: $\lambda = 2$. Then $B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ and the solution is $x = y$, with y arbitrary.

Case 3: $\lambda = 4$. Then $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and the solution is $x = -y$, with y arbitrary.

8.

$$\begin{aligned} \begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix} R_1 &\to \frac{1}{3}R_1 \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 5 & -1 & 1 & -1 \end{bmatrix} \\ R_2 &\to R_2 - 5R_1 \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} \end{bmatrix} \\ R_2 &\to \frac{-3}{8}R_2 \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{4} & 1 \end{bmatrix} \\ R_1 &\to R_1 - \frac{1}{3}R_2 \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{4} & 1 \end{bmatrix}. \end{aligned}$$

Hence the solution of the associated homogeneous system is

$$x_1 = -\frac{1}{4}x_3,\ x_2 = -\frac{1}{4}x_3 - x_4,$$

with x_3 and x_4 arbitrary.

9.

$$A = \begin{bmatrix} 1-n & 1 & \cdots & 1 \\ 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1-n \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_n \\ R_2 \to R_2 - R_n \\ \vdots \\ R_{n-1} \to R_{n-1} - R_n \end{matrix} \begin{bmatrix} -n & 0 & \cdots & n \\ 0 & -n & \cdots & n \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1-n \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \cdots & -1 \\ 0 & 1 & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1-n \end{bmatrix} R_n \to R_n - R_{n-1} \cdots - R_1 \begin{bmatrix} 1 & 0 & \cdots & -1 \\ 0 & 1 & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The last matrix is in reduced row–echelon form.

Consequently the homogeneous system with coefficient matrix A has the solution

$$x_1 = x_n,\ x_2 = x_n, \ldots, x_{n-1} = x_n,$$

with x_n arbitrary.

Alternatively, writing the system in the form

$$\begin{array}{rcl} x_1 + \cdots + x_n & = & nx_1 \\ x_1 + \cdots + x_n & = & nx_2 \\ & \vdots & \\ x_1 + \cdots + x_n & = & nx_n \end{array}$$

shows that any solution must satisfy $nx_1 = nx_2 = \cdots = nx_n$, so $x_1 = x_2 = \cdots = x_n$. Conversely if $x_1 = x_n, \ldots, x_{n-1} = x_n$, we see that $x_1, \ldots, x_n$ is a solution.

10. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and assume that $ad - bc \neq 0$.

Case 1: $a \neq 0$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} R_1 \to \tfrac{1}{a}R_1 \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} R_2 \to R_2 - cR_1 \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix}$$

$$R_2 \to \tfrac{a}{ad-bc}R_2 \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix} R_1 \to R_1 - \tfrac{b}{a}R_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Case 2: $a = 0$. Then $bc \neq 0$ and hence $c \neq 0$.

$$A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \to \begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So in both cases, A has reduced row–echelon form equal to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

11. We simplify the augmented matrix of the system using row operations:

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \end{array} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$$

$$\begin{array}{r} R_3 \to R_3 - R_2 \\ R_2 \to \frac{-1}{7}R_2 \\ R_1 \to R_1 - 2R_2 \end{array} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix} R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}.$$

Denote the last matrix by B.

Case 1: $a^2 - 16 \neq 0$. i.e. $a \neq \pm 4$. Then

$$\begin{array}{c} R_3 \to \frac{1}{a^2-16} R_3 \\ R_1 \to R_1 - R_3 \\ R_2 \to R_2 + 2R_3 \end{array} \left[\begin{array}{cccc} 1 & 0 & 0 & \frac{8a+25}{7(a+4)} \\ 0 & 1 & 0 & \frac{10a+54}{7(a+4)} \\ 0 & 0 & 1 & \frac{1}{a+4} \end{array}\right]$$

and we get the unique solution

$$x = \frac{8a+25}{7(a+4)},\ y = \frac{10a+54}{7(a+4)},\ z = \frac{1}{a+4}.$$

Case 2: $a = -4$. Then $B = \left[\begin{array}{cccc} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & 0 & -8 \end{array}\right]$, so our system is inconsistent.

Case 3: $a = 4$. Then $B = \left[\begin{array}{cccc} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$. We read off that the system is consistent, with complete solution $x = \frac{8}{7} - z,\ y = \frac{10}{7} + 2z$, where z is arbitrary.

12. We reduce the augmented array of the system to reduced row–echelon form:

$$\left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad R_3 \to R_3 + R_1 \quad \left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 \to R_3 + R_2 \quad \left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{c} R_1 \to R_1 + R_4 \\ R_3 \leftrightarrow R_4 \end{array} \quad \left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The last matrix is in reduced row–echelon form and we read off the solution of the corresponding homogeneous system:

$$\begin{aligned} x_1 &= -x_4 - x_5 = x_4 + x_5 \\ x_2 &= -x_4 - x_5 = x_4 + x_5 \\ x_3 &= -x_4 = x_4, \end{aligned}$$

where x_4 and x_5 are arbitrary elements of $\mathbb{Z}_2$. Hence there are four solutions:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}.$$

13. (a) We reduce the augmented matrix to reduced row–echelon form:

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 1 & 4 & 1 \\ 3 & 1 & 2 & 0 \end{bmatrix} \quad R_1 \to 3R_1 \quad \begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 1 & 4 & 1 \\ 3 & 1 & 2 & 0 \end{bmatrix}$$

$$\begin{array}{c} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 2R_1 \end{array} \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 4 & 3 & 3 \\ 0 & 2 & 0 & 4 \end{bmatrix} \quad R_2 \to 4R_2 \quad \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$$\begin{array}{c} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 + 3R_2 \end{array} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{c} R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 + 3R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Consequently the system has the unique solution $x = 1$, $y = 2$, $z = 0$.

(b) Again we reduce the augmented matrix to reduced row–echelon form:

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 1 & 4 & 1 \\ 1 & 1 & 0 & 3 \end{bmatrix} \quad R_1 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 1 & 0 & 3 \\ 4 & 1 & 4 & 1 \\ 2 & 1 & 3 & 4 \end{bmatrix}$$

$$\begin{array}{c} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 3R_1 \end{array} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 4 & 4 \\ 0 & 4 & 3 & 3 \end{bmatrix} \quad R_2 \to 3R_2 \quad \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 4 & 3 & 3 \end{bmatrix}$$

$$\begin{array}{c} R_1 \to R_1 + 4R_2 \\ R_3 \to R_3 + R_2 \end{array} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We read off the complete solution

$$\begin{aligned} x &= 1 - 3z = 1 + 2z \\ y &= 2 - 2z = 2 + 3z, \end{aligned}$$

where z is an arbitrary element of $\mathbb{Z}_5$.

14. Suppose that $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ are solutions of the system of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad 1 \le i \le m.$$

Then

$$\sum_{j=1}^{n} a_{ij} \alpha_j = b_i \quad \text{and} \quad \sum_{j=1}^{n} a_{ij} \beta_j = b_i$$

for $1 \le i \le m$.

Let $\gamma_i = (1-t)\alpha_i + t\beta_i$ for $1 \le i \le m$. Then $(\gamma_1, \ldots, \gamma_n)$ is a solution of the given system. For

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} \gamma_j &= \sum_{j=1}^{n} a_{ij} \{(1-t)\alpha_j + t\beta_j\} \\ &= \sum_{j=1}^{n} a_{ij}(1-t)\alpha_j + \sum_{j=1}^{n} a_{ij} t \beta_j \\ &= (1-t)b_i + tb_i \\ &= b_i. \end{aligned}$$

15. Suppose that $(\alpha_1, \ldots, \alpha_n)$ is a solution of the system of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad 1 \le i \le m. \tag{2}$$

Then the system can be rewritten as

$$\sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} a_{ij} \alpha_j, \quad 1 \le i \le m,$$

or equivalently

$$\sum_{j=1}^{n} a_{ij}(x_j - \alpha_j) = 0, \quad 1 \le i \le m.$$

So we have

$$\sum_{j=1}^{n} a_{ij} y_j = 0, \quad 1 \le i \le m.$$

where $x_j - \alpha_j = y_j$. Hence $x_j = \alpha_j + y_j$, $1 \le j \le n$, where $(y_1, \ldots, y_n)$ is a solution of the associated homogeneous system. Conversely if $(y_1, \ldots, y_n)$

is a solution of the associated homogeneous system and $x_j = \alpha_j + y_j,\ 1 \le j \le n$, then reversing the argument shows that $(x_1, \ldots, x_n)$ is a solution of the system 2 .

16. We simplify the augmented matrix using row operations, working towards row–echelon form:

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ a & 1 & 1 & 1 & b \\ 3 & 2 & 0 & a & 1+a \end{bmatrix} \begin{array}{c} R_2 \to R_2 - aR_1 \\ R_3 \to R_3 - 3R_1 \end{array} \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1-a & 1+a & 1-a & b-a \\ 0 & -1 & 3 & a-3 & a-2 \end{bmatrix}$$

$$\begin{array}{c} R_2 \leftrightarrow R_3 \\ R_2 \to -R_2 \end{array} \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 3-a & 2-a \\ 0 & 1-a & 1+a & 1-a & b-a \end{bmatrix}$$

$$R_3 \to R_3 + (a-1)R_2 \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 3-a & 2-a \\ 0 & 0 & 4-2a & (1-a)(a-2) & -a^2+2a+b-2 \end{bmatrix} = B.$$

Case 1: $a \ne 2$. Then $4 - 2a \ne 0$ and

$$B \to \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 3-a & 2-a \\ 0 & 0 & 1 & \frac{a-1}{2} & \frac{-a^2+2a+b-2}{4-2a} \end{bmatrix}.$$

Hence we can solve for x, y and z in terms of the arbitrary variable w.

Case 2: $a = 2$. Then

$$B = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & b-2 \end{bmatrix}.$$

Hence there is no solution if $b \ne 2$. However if $b = 2$, then

$$B = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and we get the solution $x = 1 - 2z,\ y = 3z - w$, where w is arbitrary.

17. (a) We first prove that $1 + 1 + 1 + 1 = 0$. Observe that the elements

$$1+0, \quad 1+1, \quad 1+a, \quad 1+b$$

are distinct elements of F by virtue of the *cancellation law for addition.* For this law states that $1+x = 1+y \Rightarrow x = y$ and hence $x \neq y \Rightarrow 1+x \neq 1+y$.

Hence the above four elements are just the elements 0, 1, a, b in some order. Consequently

$$\begin{aligned}(1+0)+(1+1)+(1+a)+(1+b) &= 0+1+a+b \\ (1+1+1+1)+(0+1+a+b) &= 0+(0+1+a+b),\end{aligned}$$

so $1+1+1+1 = 0$ after cancellation.

Now $1+1+1+1 = (1+1)(1+1)$, so we have $x^2 = 0$, where $x = 1+1$. Hence $x = 0$. Then $a+a = a(1+1) = a \cdot 0 = 0$.

Next $a+b = 1$. For $a+b$ must be one of 0, 1, a, b. Clearly we can't have $a+b = a$ or b; also if $a+b = 0$, then $a+b = a+a$ and hence $b = a$; hence $a+b = 1$. Then

$$a+1 = a+(a+b) = (a+a)+b = 0+b = b.$$

Similarly $b+1 = a$. Consequently the addition table for F is

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

.

We now find the multiplication table. First, ab must be one of 1, a, b; however we can't have $ab = a$ or b, so this leaves $ab = 1$.

Next $a^2 = b$. For a^2 must be one of 1, a, b; however $a^2 = a \Rightarrow a = 0$ or $a = 1$; also

$$a^2 = 1 \Rightarrow a^2 - 1 = 0 \Rightarrow (a-1)(a+1) = 0 \Rightarrow (a-1)^2 = 0 \Rightarrow a = 1;$$

hence $a^2 = b$. Similarly $b^2 = a$. Consequently the multiplication table for F is

×	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

.

(b) We use the addition and multiplication tables for F:

$$A = \begin{bmatrix} 1 & a & b & a \\ a & b & b & 1 \\ 1 & 1 & 1 & a \end{bmatrix} \begin{array}{c} R_2 \to R_2 + aR_1 \\ R_3 \to R_3 + R_1 \end{array} \begin{bmatrix} 1 & a & b & a \\ 0 & 0 & a & a \\ 0 & b & a & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & a & b & a \\ 0 & b & a & 0 \\ 0 & 0 & a & a \end{bmatrix} \begin{matrix} R_2 \to aR_2 \\ R_3 \to bR_3 \end{matrix} \quad \begin{bmatrix} 1 & a & b & a \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_1 + aR_2 \quad \begin{bmatrix} 1 & 0 & a & a \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 + aR_3 \\ R_2 \to R_2 + bR_3 \end{matrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The last matrix is in reduced row–echelon form.