DETERMINANT4

LEMMA 0.1. *Let $\vartheta > 7$ be a real number, $P, R, \in \mathbb{Z}, Q, S \in \mathbb{N}$ and $|PS - QR| = 4$. Then P/Q is a convergent to $\theta = (P\vartheta + R)/(Q\vartheta + S)$.*

Proof. Let $P/Q = [u_0, \ldots, u_n]$, where n is chosen so that $PS - QR = (-1)^{n+1}4$. Let $p_{-2}, p_{-1}, p_0, \ldots$ and $q_{-2}, q_{-1}, q_0, \ldots$ be the numerators and denominators of the continued fraction. Then $P = p_n, Q = q_n$ and

$$p_nS - q_nR = PS - QR = (-1)^{n+1}4 = 4(p_nq_{n-1} - p_{n-1}q_n).$$

Hence $p_n(S - 4q_{n-1}) = q_n(R - 4p_{n-1})$ and we have $S - 4q_{n-1} = kq_n$ for some $k \in \mathbb{Z}$. We assert that $k \geq -3$. First let $n = 0$. Then $S = k \geq 1$, as $q_{-1} = 0$ and $q_0 = 1$. So we assume $n \geq 1$. If $4Q < S$, then $kq_n = S - 4q_{n-1} > 4(q_n - q_{n-1}) \geq 0$, and hence $k \geq 0$.

If $4Q \geq S$, then

$$-4q_n \leq -4q_{n-1} < S - 4q_{n-1} \leq 4(q_n - q_{n-1}) < 4q_n.$$

Hence $|k|q_n = |S - 4q_{n-1}| < 4q_n$ and so $|k| \leq 3$. Also note that $R - 4p_{n-1} = kp_n$. Hence

$$\theta = \frac{P\vartheta + R}{Q\vartheta + S} = \frac{p_n(\vartheta + k) + 4p_{n-1}}{q_n(\vartheta + k) + 4q_{n-1}} = \frac{p_n(\vartheta + k)/4 + p_{n-1}}{q_n(\vartheta + k)/4 + q_{n-1}}.$$

However as $k \geq -3$, we have $(\vartheta + k)/4 \geq (\vartheta - 3)/4 > 1$ if $\vartheta > 7$. Hence p_n/q_n is a convergent to θ,

□

REMARK 0.1. We apply this when $\vartheta = \sqrt{D}$ and $D \equiv 5 \pmod 8$.

Date: August 13, 2016.