An algorithm for deciding if ξ is NSCF reduced.

Keith R. Matthews

1. Introduction

Suppose ξ_0 is NSCF reduced. Then we know ξ_0 or $\xi_0 - 1$ is RCF reduced, say η_0. Then an algorithm of Selenius (Theorem 1, [2]) shows that the NSCF expansion of ξ_0 is obtained from the positive–negative representations of the η_i in jumps of 1 (with corresponding NSCF complete quotient η_{i+1}) or 2 (with corresponding NSCF complete quotient $\eta_{i+2} + 1$. So we can picture the RCF complete quotients as lying on a circle, with the NSCF complete quotients ξ_r or $\xi_r + 1$ located as a subsequence of the RCF complete quotients.

If ξ_0 is NSCF reduced, we know there exists a NSCF reduced ξ_{-2}. This will be a special surd and either ξ_{-2} or $\xi_{-2}-1$ will be either η_{-2}, η_{-3} or η_{-4}, corresponding to the four possible jump sequences: $1,1$; $2,1$; $1,2$; $2,2$. These are listed in the table below.

The algorithm was superceded by a more elegant one in [3].

2. The algorithm

Case 0. If neither ξ nor $\xi - 1$ is RCF reduced, output ξ is not NSCF reduced.

$\xi = \eta_0$		$\xi - 1 = \eta_0$	
Case 1(a)	Case 1(b)	Case 2(a)	Case 2(b)
η_{-2} η_{-1} η_0	η_{-3} η_{-2} η_{-1} η_0	η_{-3} η_{-2} η_{-1} η_0	η_{-4} η_{-3} η_{-2} η_{-1} η_0

Case 1: $\xi = \eta_0$ is RCF reduced.

(a) Find η_{-2}, get its NSCF successor x say. If $\epsilon = -1$, go to Case 1(b); otherwise test η_{-2} to see if it is special. If not, go to Case 1(b); otherwise get the NSCF successor of x - say y and check that it's equal to ξ. If not, go to Case 1(b); otherwise output that ξ is NSCF reduced and x is a semi-reduced predecessor of ξ.

Date: October 11, 2010.

(b) Find η_{-3}, get its NSCF successor x say. If $\epsilon = 1$, output ξ is not NSCF reduced; otherwise test η_{-3} to see if it is special. If not, output ξ is not NSCF reduced; otherwise get the NSCF successor of x - say y and check that it's equal to ξ. If not, output ξ is not NSCF reduced; otherwise output that ξ is NSCF reduced and x is a semi-reduced predecessor of ξ.

Case 2: $\xi - 1 = \eta_0$ is RCF reduced.

(a) Find η_{-3}, get its NSCF successor x say. If $\epsilon = -1$, go to case 2(b); otherwise test η_{-4} to see if it is special. If not, go to case 2(b); otherwise get the NSCF successor of x - say y and check that it's equal to ξ. If not, go to Case 2(b); otherwise output that ξ is NSCF reduced and x is a semi-reduced predecessor of ξ.

(b) Find η_{-4}, get its NSCF successor x say. If $\epsilon = 1$, output that ξ is not NSCF reduced; otherwise test η_{-4} to see if it is special. If not, output that ξ is not NSCF reduced; otherwise get the NSCF successor of x - say y and check that it's equal to ξ. If not, output that ξ is not NSCF reduced; otherwise output that ξ is NSCF reduced and x is a semi-reduced predecessor of ξ.

We know by Theorem IX, [1] that x is in fact NSCF–reduced, unless x has the form $\frac{p+q+\sqrt{p^2+q^2}}{2q}$, where $p > 2q > 0$, in which case $x - 1 = \frac{p-q+\sqrt{p^2+q^2}}{2q}$ is the unique NSCF–reduced predecessor of ξ. Note that this can only occur in Cases 1(b) or 2(b), as $\frac{p+q+\sqrt{p^2+q^2}}{2q}$ is not RCF–reduced and hence cannot result from an initial jump of 1, as would be the case in Cases 1(a) and 1(c).

3. Examples

$\xi_0 = \frac{16+\sqrt{624}}{23}, \frac{16+\sqrt{624}}{16}, \frac{30+\sqrt{624}}{12}, \frac{30+\sqrt{624}}{23}$ are examples of cases 1(a), 1(b), 2(a), 2(b) respectively.

References

[1] A. A. K. Ayyangar, Theory of the nearest square continued fraction, *J. Mysore Univ.* Sect. A. **1** (1941) 21–32, 97–117.

[2] K.R. Matthews and J.P. Robertson, Period–length equality for the nearest integer and nearest square continued fraction expansions of a quadratic surd, mo appear, Math. Glasnik, December 2011.

[3] K.R. Matthews and J.P. Robertson, On purely periodic nearest square continued fractions, submitted.