ON THE CONVERGENTS OF SEMI–REGULAR CONTINUED FRACTIONS

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1. Introduction

On page 161 of Perron's book [7], it is proved that a convergent of the nearest integer continued fraction (NICF) is also a convergent of the regular continued fraction (RCF). We give another proof, which generalises to semi–regular continued fractions. We also give an application to the nearest square continued fraction. Our method is based on the following result, which is Theorem 172, [3, pp. 140-141]:

Lemma 1. *If* $\omega = \frac{P\zeta+R}{Q\zeta+S}$*, where* $\zeta > 1$ *and* P, Q, R, S *are integers such that* $Q > S > 0$ *and* $PS - QR = \pm 1$*, then* P/Q *is an RCF convergent* A_n/B_n *to* ω *and* $R/S = A_{n-1}/B_{n-1}$*. Also* $\zeta = \zeta_{n+1}$*, the* $(n+1)$*–th RCF complete convergent to* ω*.*

2. Nearest integer continued fractions

We use the following notation for the nearest integer expansion:

$$\xi_0 = \tilde{a}_0 + \frac{\epsilon_1|}{|\tilde{a}_1} + \cdots + \frac{\epsilon_n|}{|\tilde{a}_n} + \cdots, \tag{2.1}$$

with $\tilde{\xi}_n$ the n–th complete quotient and $\tilde{A}_n/\tilde{B}_n$ the n–th convergent. We remark that for $n \geq 1$,

$$\tilde{a}_n \geq 2, \tag{2.2}$$

$$\tilde{\xi}_n > 2. \tag{2.3}$$

Then (2.2) implies that $\tilde{B}_n > \tilde{B}_{n-1} \geq 1$ for $n \geq 1$.

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Theorem 1. *For the nearest integer continued fraction* (2.1)

(i) *if* $\epsilon_{n+1} = 1$*, then* $\tilde{\xi}_{n+1} = \xi_k$*, where*

$$\tilde{A}_n/\tilde{B}_n = A_{k-1}/B_{k-1} \text{ and } \tilde{A}_{n-1}/\tilde{B}_{n-1} = A_{k-2}/B_{k-2}.$$

(ii) *if* $\epsilon_{n+1} = -1$*, then* $\tilde{\xi}_{n+1} = \xi_k + 1$*, where*

$$\tilde{A}_n/\tilde{B}_n = A_{k-1}/B_{k-1} \text{ and } (\tilde{A}_n - \tilde{A}_{n-1})/(\tilde{B}_n - \tilde{B}_{n-1}) = A_{k-2}/B_{k-2}.$$

Proof. We consider the following equation (Perron [7, p. 19]):

$$\xi_0 = \frac{\tilde{A}_n\tilde{\xi}_{n+1} + \epsilon_{n+1}\tilde{A}_{n-1}}{\tilde{B}_n\tilde{\xi}_{n+1} + \epsilon_{n+1}\tilde{B}_{n-1}}. \tag{2.4}$$

If $\epsilon_{n+1} = 1$, then by Lemma 1, as $\tilde{\xi}_{n+1} > 1, \Delta_n = \tilde{A}_n\tilde{B}_{n-1} - \tilde{B}_n\tilde{A}_{n-1} = \pm 1$ and $\tilde{B}_n > \tilde{B}_{n-1} > 0$, it follows that for some k, $\tilde{\xi}_{n+1} = \xi_k$ and $\tilde{A}_n/\tilde{B}_n = A_{k-1}/B_{k-1}$, $\tilde{A}_{n-1}/\tilde{B}_{n-1} = A_{k-2}/B_{k-2}$.

If $\epsilon_{n+1} = -1$, then

$$\xi_0 = \frac{\tilde{A}_n(\tilde{\xi}_{n+1} - 1) + \tilde{A}_n - \tilde{A}_{n-1}}{\tilde{B}_n(\tilde{\xi}_{n+1} - 1) + \tilde{B}_n - \tilde{B}_{n-1}}. \tag{2.5}$$

Since $\tilde{\xi}_{n+1} > 2$ and $\tilde{A}_n(\tilde{B}_n - \tilde{B}_{n-1}) - \tilde{B}_n(\tilde{A}_n - \tilde{A}_{n-1}) = -\Delta_n = \pm 1$, we deduce, again by Lemma 1, that for some k, $\tilde{\xi}_{n+1} - 1 = \xi_k$ and that $\tilde{A}_n/\tilde{B}_n = A_{k-1}/B_{k-1}$ and $(\tilde{A}_n - \tilde{A}_{n-1})/(\tilde{B}_n - \tilde{B}_{n-1}) = A_{k-2}/B_{k-2}$. □

3. Semi–regular continued fractions

We now generalize this result to semi–regular continued fractions. We need the following lemmas.

Lemma 2. *If* $\omega = \frac{P\zeta + R}{Q\zeta + S}$*, where* $\zeta > 1$ *and* P, Q, R, S *are integers such that* $Q > 0, S > 0$ *and* $PS - QR = \pm 1$*, or* $S = 0$ *and* $Q = 1 = R$*, then* P/Q *is a convergent* A_n/B_n *to* ω*.*

Proof. This is an extension of Theorem 172, Hardy and Wright ([3, pp. 140—141]), who dealt with the case $Q > S > 0$. See Matthews [4, pp. 325–326]. □

Lemma 3. *Let*

$$\xi_0 = \tilde{a}_0 + \frac{\epsilon_1}{\lceil \tilde{a}_1} + \cdots + \frac{\epsilon_n}{\lceil \tilde{a}_n} + \cdots, \tag{3.1}$$

denote a semi–regular continued fraction expansion, with n–th complete quotient $\tilde{\xi}_n$ and n–th convergent $\tilde{A}_n/\tilde{B}_n$. Then for $n \geq 0$,

$$\tilde{B}_n \geq 1, \tag{3.2}$$

$$\epsilon_{n+1} = -1 \implies \tilde{B}_n > \tilde{B}_{n-1}. \tag{3.3}$$

Remark. If $\xi_0 = (133 + \sqrt{722})/361$, $\tilde{B}_1 = 3 > \tilde{B}_2 = 2$ and $\epsilon_3 = 1$.

Proof. (3.2) follows from Satz 5.1, [7, p. 135], while (3.3) follows from Satz 1, [2, p. 10]. Alternatively, see Lemma 1, [6]. □

Noting that $\tilde{\xi}_{n+1} > 1$ holds for a semi–regular continued fraction, the proof of Theorem 1 then generalizes.

Theorem 2. *For the semi–regular continued fraction* (3.1)

(i) *If $\epsilon_{n+1} = 1$, then $\tilde{\xi}_{n+1} = \xi_k$, where $\tilde{A}_n/\tilde{B}_n = A_{k-1}/B_{k-1}$.*

(ii) *If $\epsilon_{n+1} = -1$ and $\tilde{\xi}_{n+1} > 2$, then $\tilde{\xi}_{n+1} = \xi_k + 1$, where*

$$\tilde{A}_n/\tilde{B}_n = A_{k-1}/B_{k-1} \text{ and } (\tilde{A}_n - \tilde{A}_{n-1})/(\tilde{B}_n - \tilde{B}_{n-1}) = A_{k-2}/B_{k-2}.$$

4. Nearest square continued fractions

The NSCF is an example of a semi–regular continued fraction. In Theorem 2, we can remove the restriction $\tilde{\xi}_{n+1} > 2$, if $\tilde{\xi}_n$ is NSCF–reduced.

Lemma 4. *If $\tilde{\xi}_n$ is NSCF–reduced and $\epsilon_{n+1} = -1$, then $\tilde{\xi}_{n+1} > 2$.*

Proof. If $\xi_n = \frac{\tilde{P}_n + \sqrt{D}}{\tilde{Q}_n}$ is NSCF–reduced, from Ayyangar [1, p. 22], we have $\tilde{Q}_{n+1}^2 + \frac{1}{4}\tilde{Q}_n^2 \leq D$, so $|\tilde{Q}_{n+1}| < \sqrt{D}$. Also $\tilde{\xi}_n$ is the successor of a special surd and so by Theorem 1(iv), Ayyangar [1, p. 22], $\tilde{Q}_n > 0$. Similarly $\tilde{Q}_{n+1} > 0$. Moreover, by Theorem 1(i), [1, p. 22], $\epsilon_{n+1} = -1$ implies $\tilde{P}_{n+1} \geq \tilde{Q}_{n+1} + \frac{1}{2}\tilde{Q}_n$. Hence

$$\tilde{\xi}_{n+1} = \frac{\tilde{P}_{n+1} + \sqrt{D}}{\tilde{Q}_{n+1}} \geq \frac{\tilde{Q}_{n+1} + \frac{1}{2}\tilde{Q}_n + \sqrt{D}}{\tilde{Q}_{n+1}} > \frac{\tilde{Q}_{n+1} + \sqrt{D}}{\tilde{Q}_{n+1}} > \frac{2\tilde{Q}_{n+1}}{\tilde{Q}_{n+1}} = 2.$$

□

Remark. The NSCF expansion of $\xi_0 = (133 + \sqrt{722})/361$ is an example where $\tilde{A}_1/\tilde{B}_1 = 1/3$ is not a convergent of the RCF expansion of ξ_0. Here $\epsilon_2 = -1$. However $\tilde{\xi}_2 = (-8 + \sqrt{722})/14 < 2$, so we cannot apply Theorem 2 (ii).

References

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