RCF TO NICF

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ABSTRACT. Selenius gave an algorithm for converting the regular continued fraction (RCF) of $\sqrt{D}$ into the nearest square continued fraction (NSCF). We give an similar result for converting the RCF into the nearest integer continued fraction (NICF).

1. The starting observation

Lemma 1. *Let $\xi_0 = (a_0, a_1, \ldots)$ be an RCF expansion. Then if $[x]$ denoted the nearest integer to x, we have*

$$a_n = [\xi_n] \text{ if an only if } a_{n+1} > 1.$$

Proof.

$$\xi_n = a_n + \frac{1}{\xi_{n+1}}, \quad \xi_{n+1} > 1.$$

(a) Assume $a_{n+1} = 1$. Then

$$\xi_n - a_n = \frac{1}{1+\frac{1}{\xi_{n+2}}} = \frac{\xi_{n+2}}{\xi_{n+2}+1}, \quad \xi_{n+2} > 1$$

$$a_n + 1 - \xi_n = 1 - \frac{1}{1+\frac{1}{\xi_{n+2}}} = \frac{1}{\xi_{n+2}}.$$

Hence

$$a_n + 1 - \xi_n = \frac{1}{\xi_{n+2}} < \frac{\xi_{n+2}}{\xi_{n+2}+1} = \xi_n - a_n$$

and so $a_n + 1 = [\xi_n]$.

(b) Assume $a_{n+1} > 1$. Then

$$\xi_n - a_n = \frac{\xi_{n+1}}{a_{n+1}\xi_{n+1}+1}$$

$$a_n + 1 - \xi_n = 1 - \frac{\xi_{n+1}}{a_{n+1}\xi_{n+1}+1} = \frac{(a_{n+1}-1)\xi_{n+1}+1}{a_{n+1}\xi_{n+1}+1}.$$

Hence

$$\xi_n - a_n = \frac{\xi_{n+1}}{a_{n+1}\xi_{n+1}+1} < \frac{(a_{n+1}-1)\xi_{n+1}+1}{a_{n+1}\xi_{n+1}+1} = a_n + 1 - \xi_n$$

and so $a_n = [\xi_n]$.

□

2. Selenius' Lemma

In [3, p.62], Selenius proved the following:

Lemma 2. *Let ξ_n be a complete quotient for the RCF expansion of $\sqrt{D}$, with positive and negative representations for $\nu \geq 1$:*

$$\xi_{\nu-1} = \frac{P_{\nu-1}+\sqrt{D}}{Q_{\nu-1}} = b_{\nu-1} + \frac{Q_\nu}{P_\nu+\sqrt{D}}. = b_{\nu-1}+1-\frac{Q_\nu^{''}}{P_\nu^{''}+\sqrt{D}}, \tag{2.1}$$

where $b_{\nu-1} = \lfloor \xi_{\nu-1} \rfloor$. Then

(1) *If $b_\nu = 1$, then*
   (a) $Q_\nu^{''} = Q_{\nu+1}$ *and conversely;*
   (b) $P_\nu^{''} = P_{\nu+1} + Q_{\nu+1}$.
(2) *If $b_\nu \geq 2$, then $Q_\nu^{''} - Q_\nu > 0$ (thus $Q_\nu^{''} \leq Q_\nu$ implies $b_\nu = 1$).*

A study of the proof of the lemma reveals that it remains true if ξ_0 is a reduced quadratic irrational or else $Q_0 > 0$ and ξ_1 is a reduced quadratic irrational.

Selenius then shows how the lemma gives an algorithm for converting the RCF of $\sqrt{D}$ to its NSCF.

Let $\overline{\xi}_0, \overline{\xi}_1, \ldots,$ denote the complete NSCF convergents of $\overline{\xi}_0 = \xi_0$. Then if $Q_1 < Q_1^{''}$, we choose the positive representation $\overline{\xi}_1 = \xi_1$, with $\overline{b}_0 = b_0$ and $\epsilon_0 = 1$.

If $Q_1 \geq Q_1^{''}$, we choose the negative representation

$$\overline{\xi}_1 = \frac{P_1^{''}+\sqrt{D}}{Q_1^{''}}, \overline{b}_0 = b_o + 1, \epsilon_0 = -1.$$

But by the Lemma, we have $b_1 = 1$, $Q_1^{''} = Q_2$ and $P_1^{''} = P_2 + Q_2$. Hence

$$\begin{aligned}\overline{\xi}_1 &= \frac{P_2+Q_2+\sqrt{D}}{Q_2} = \xi_2 + 1\\ &= b_2 + 1 + \frac{Q_3}{P_3+\sqrt{D}}. = b_2 + 2 - \frac{Q_3^{''}}{P_3^{''}+\sqrt{D}}\end{aligned}$$

and $\overline{b}_1 = b_2 + 1$ or $b_2 + 2$. Thus to determine $\overline{\xi}_1$, we have to skip over the representation $\xi_1 = 1 + \frac{Q_2}{P_2+\sqrt{D}}$.

3. Our supplement to Selenius' lemma

We give a result that gives rise to an algorithm for converting the RCF to the NICF, similar to the one mentioned above by Selenius.

Lemma 3. *Suppose $Q_{\nu-1} > 0$. Then*

$$Q_\nu + Q_\nu^{''} < 2\sqrt{D} \text{ if and only if } b_\nu = 1. \tag{3.1}$$

Proof.

$$\begin{aligned}
b_\nu = 1 &\iff b_{\nu-1} = [\xi_{\nu-1}] \\
&\iff \frac{Q_\nu}{P_\nu + \sqrt{D}} > \frac{Q''}{P''_\nu + \sqrt{D}} \\
&\iff \frac{Q_\nu(P_\nu - \sqrt{D})}{{P_\nu}^2 - D} > \frac{Q''(P''_\nu - \sqrt{D})}{{P''_\nu}^2 - D} \\
&\iff \frac{Q_\nu(P_\nu - \sqrt{D})}{-Q_{\nu-1}Q_\nu} > \frac{Q''(P''_\nu - \sqrt{D})}{Q_{\nu-1}Q''_\nu} \\
&\iff \sqrt{D} - P_\nu > P''_\nu - \sqrt{D} \\
&\iff 2\sqrt{D} > P_\nu + P''_\nu \\
&\iff 2\sqrt{D} > Q_\nu + Q''_\nu.
\end{aligned}$$

□

4. RCF → NICF

Selenius' algorithm, with the condition $Q''_\nu \leq Q_\nu$ replaced by $Q_\nu + Q''_\nu < 2\sqrt{D}$, enables us to convert the RCF of standard form $\xi_0 = (u + \sqrt{d})/v$ to the NICF. It is assusmed that either ξ_0 is reduced, or $Q_0 > 0$ and ξ_1 is reduced.

$k = 1$
$flag = 0$
$jj = -1$
$f = \lfloor\sqrt{d}\rfloor$
$P_0 = u, Q_0 = v$
for $(j \geq 0)\{$
$a_j = \lfloor\frac{f+P_j}{Q_j}\rfloor$
$P_{j+1} = a_jQ_j - P_j$
$Q_{j+1} = (d - P_{j+1}^2)/Q_j$
$P_{j+1}^{''} = P_{j+1} + Q_j$
$Q_{j+1}^{''} = P_{j+1}^{''} + P_{j+1} - Q_{j+1}$
if $(flag = 0$ or $j = jj)\{$
if $(4d > (Q_{j+1} + Q_{j+1}^{''})^2)\{$
$r = 1$
else:
$r = 0$
$\}$
if $(j = jj)\{$
$a = a_j + 1$
$flag = 0$
$\}$
if $(r = 0)\{$
$\overline{P}_k = P_{j+1}, \overline{Q}_k = Q_{j+1}, b_{k-1} = a, e_k = 1$
$flag = 0$
$\}$
if $(r = 1)\{$
$\overline{P}_k = P_{j+1}^{''}, \overline{Q}_k = Q_{j+1}^{''}, b_{k-1} = a + 1, e_k = -1$
$flag = 1$
$jj = j + 2$
$\}$
$k = k + 1$
$\}$
$j = j + 1$
$\}$

As with Selenius, we illustrate the algorithm with the example $\xi_0 = \sqrt{97}$. In what follows, (P, Q) denotes $\frac{P+\sqrt{97}}{Q}$, $\xi_j = (P_j + \sqrt{97})/Q_j$, $\overline{\xi}_k = (\overline{P}_k + \sqrt{97})/\overline{Q}_k$.

Here is the RCF output as far as the end of the first period:

$$
\begin{aligned}
\xi_0 &= (0,1) = 9 + (9,16)^{-1} = 10 - (10,3)^{-1}\\
\xi_1 &= (9,16) = 1 + (7,3)^{-1} = 2 - (23,27)^{-1}\\
\xi_2 &= (7,3) = 5 + (8,11)^{-1} = 6 - (11,8)^{-1}\\
\xi_3 &= (8,11) = 1 + (3,8) = 2 - (14,9)\\
\xi_4 &= (3,8) = 1 + (5,9)^{-1} = 2 - (13,9)^{-1}\\
\xi_5 &= (5,9) = 1 + (4,9)^{-1} = 2 - (13,8)^{-1}\\
\xi_6 &= (4,9) = 1 + (5,8)^{-1} = 2 - (14,11)^{-1}\\
\xi_7 &= (5,8) = 1 + (3,11)^{-1} = 2 - (11,3)^{-1}\\
\xi_8 &= (3,11) = 1 + (8,3)^{-1} = 2 - (19,24)^{-1}\\
\xi_9 &= (8,3) = 5 + (7,16)^{-1} = 6 - (10,1)^{-1}\\
\xi_{10} &= (7,16) = 1 + (9,1)^{-1} = 2 - (25,33)^{-1}\\
\xi_{11} &= (9,1) = 18 + (9,16)^{-1} = 19 - (10,3)^{-1}\\
\xi_{12} &= \xi_1.
\end{aligned}
$$

Here is the resulting NICF expansion:

$$
\begin{aligned}
\overline{\xi}_0 &= (0,1) = 9 + (9,16)^{-1} = 10 - (10,3)^{-1}\\
\overline{\xi}_1 &= (10,3) = 6 + (8,11)^{-1} = 7 - (11,8)^{-1}\\
\overline{\xi}_2 &= (11,8) = 2 + (5,9)^{-1} = 3 - (13,9)^{-1}\\
\overline{\xi}_3 &= (13,9) = 2 + (5,8)^{-1} = 3 - (14,11)^{-1}\\
\overline{\xi}_4 &= (14,11) = 2 + (8,3)^{-1} = 3 - (19,24)^{-1}\\
\overline{\xi}_5 &= (8,3) = 5 + (7,16)^{-1} = 6 - (10,1)^{-1}\\
\overline{\xi}_6 &= (10,1) = 19 + (9,16)^{-1} = 20 - (10,3)^{-1}\\
\overline{\xi}_7 &= \overline{\xi}_1.
\end{aligned}
$$

$$
\sqrt{97} = 10 - \frac{1|}{\underset{*}{|7}} - \frac{1|}{|3} - \frac{1|}{|3} - \frac{1|}{|2} + \frac{1|}{|6} - \frac{1|}{\underset{*}{|20}}
$$

References

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