PERIOD 2 NSCF AND NICF EXPANSIONS OF $\sqrt{D}$

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Abstract. We prove a conjecture of John Robertson.

Theorem 1. *If* $\sqrt{D} = \tilde{b}_0 + \frac{\epsilon_1|}{|\underset{*}{\tilde{b}_1}} + \frac{\epsilon_2|}{|\underset{*}{\tilde{b}_2}}$ *is a NSCF or NICF period* 2 *expansion, then*

$$(\epsilon_1, \epsilon_2) = (1, 1) \text{ or } (-1, -1).$$

Proof. Assume $\sqrt{D}$ has NSCF and NICF period-length 2 and that $\epsilon_1\epsilon_2 = -1$. Then the RCF expansion has period-length 3. We consider the RCF to NSCF and NICF transformations.

Case 1. $\epsilon_1 = 1, \epsilon_2 = -1$. Then $\epsilon_1 = 1$ implies $\tilde{\xi}_1 = \xi_1$ and $\epsilon_2 = -1$ implies

$$\tilde{\xi}_2 = \xi_3 + 1. \tag{0.1}$$

But $\epsilon_3 = 1$ by NSCF/NICF periodicity, so (0.1) implies

$$\tilde{b}_2 = b_3 + 1. \tag{0.2}$$

However $\tilde{b}_2 = 2b_0 = b_3$, so (0.2) gives the contradiction $2b_0 = 2b_0 + 1$.

Case 2. $\epsilon_1 = -1, \epsilon_2 = 1$. Then $\epsilon_1 = -1$ implies $\tilde{\xi}_1 = \xi_2 + 1$ and $\epsilon_2 = 1$ implies

$$\tilde{\xi}_2 = \xi_3. \tag{0.3}$$

But $\epsilon_3 = -1$ by NSCF/NICF periodicity, so (0.3) implies

$$\tilde{b}_2 = b_3 + 1 \tag{0.4}$$

and again we get a contradiction. □

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