On the continued fraction expansion of $\sqrt{2^{2n+1}}$

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Abstract

We derive limited information about the period of the continued fraction expansion of $\sqrt{2^{2n+1}}$: The period-length is a multiple of 4 if $n > 1$. Also the central norm $Q_m = 4$ and the central partial quotient $a_m = \lfloor\sqrt{2^{2n-1}}\rfloor$ or $\lfloor\sqrt{2^{2n-1}}\rfloor - 1$, whichever is odd. It seems likely that $l_n/2^n \to .7427\cdots$.

1 Introduction

Let $D_n = 2^{2n+1}$ and l_n be the length of the period of the continued fraction for $\sqrt{D_n}$.

We observe that l_n is even, as otherwise the negative Pell equation $x^2 - 2^{2n+1}y^2 = -1$ would have a solution. Here x is odd, giving the contradiction $x^2 \equiv -1 \pmod 8$.

n	The continued fraction expansion of $\sqrt{2^{2n+1}}$	l_n
0	$[1, \overline{2}]$	1
1	$[2, \overline{1, 4}]$	2
2	$[5, \overline{1, 1, 1, 10}]$	4
3	$[11, \overline{3, 5, 3, 22}]$	4
4	$[22, \overline{1, 1, 1, 2, 6, 11, 6, 2, 1, 1, 1, 44}]$	12
5	$[45, \overline{3, 1, 12, 5, 1, 1, 2, 1, 2, 4, 1, 21, 1, 4, 2, 1, 2, 1, 1, 5, 12, 1, 3, 90}]$	24

The values of l_n for $n \le 31$ are given in sequence A059927 of [6]. Don Reble communicated l_{32} to the author:

n	l_n
0	1
1	2
2	4
3	4
4	12
5	24
6	48
7	96
8	196
9	368
10	760
11	1524
12	3064
13	6068
14	12168
15	24360
16	48668
17	97160
18	194952
19	389416
20	778832
21	1557780
22	3116216
23	6229836
24	12462296
25	24923320
26	49849604
27	99694536
28	199394616
29	398783628
30	797556364
31	1595117676
32	3190297400
33	6380517544
34	12761088588
35	25522110948

We prove that l_n is a multiple of 4 if $n > 1$. Also with $l_n = 2m$, the central norm $Q_m = 4$ and the central partial quotient $a_m = \lfloor\sqrt{D_{n-1}}\rfloor$ or $\lfloor\sqrt{D_{n-1}}\rfloor - 1$, whichever is odd.

We need some facts about the least solution of the Pell equation $x^2 - 2^{2n+1}y^2 = 1$.

Let $D_n = 2^{2n+1}$ and ϵ_n denote the fundamental solution of the Pell equation $x^2 - 2^{2n+1}y^2 = 1$, ie. the solution with least positive x and y.

Then J. Schur ([5, p. 36]) gave the following formula for ϵ_n. (There was a misprint - $D' = 2^{2l+1}$ should be $D' = 2^{2l-1}$.)

Lemma 1.

$$\epsilon_n = (3+\sqrt{8})^{2^{n-1}} (= (1+\sqrt{2})^{2^n}) \qquad (1)$$

Proof. Let u_n and v_n be defined by for $n \geq 1$ by $u_1 = 3, v_1 = 1$ and

$$u_n = 2^{2n}v_{n-1}^2 + 1, v_n = u_{n-1}v_{n-1}.$$

for $n > 1$. Then we see by induction that

1. v_n is odd,
2. $u_n^2 - D_n v_n^2 = 1$ for all $n \geq 1$,
3. $u_n + v_n\sqrt{D_n} = (u_{n-1} + v_{n-1}\sqrt{D_{n-1}})^2$,
4. $u_n + v_n\sqrt{D_n} = (3+\sqrt{8})^{2^{n-1}}$.

We now prove that $\epsilon_n = u_n + v_n\sqrt{D_n}$. This true when $n = 1$. So let $n > 1$ and assume $\epsilon_{n-1} = u_{n-1} + v_{n-1}\sqrt{D_{n-1}}$.

Now assume $1 = u^2 - 2^{2n+1}v^2, u \geq 1, v \geq 1$.

Then $u^2 - 2^{2n-1}(2v)^2 = 1$, so

$$u + 2v\sqrt{D_{n-1}} = (u_{n-1} + v_{n-1}\sqrt{D_{n-1}})^i,$$

for some $i \geq 1$. But $i = 1$ would imply $2v = v_{n-1}$, contradicting the fact that v_{n-1} is odd. Also

$$(u_{n-1} + v_{n-1}\sqrt{D_{n-1}})^2 = u_{n-1}^2 + v_{n-1}^2 D_{n-1} + 2u_{n-1}v_{n-1}\sqrt{D_{n-1}}.$$

Hence $2v \geq 2u_{n-1}v_{n-1} = 2v_n$ and so $v \geq v_n$ and hence $u_n + v_n\sqrt{D_n} = \epsilon_n$.

n	ϵ_n
1	$3+\sqrt{8}$
2	$17+3\sqrt{32}$
3	$577+51\sqrt{128}$
4	$665857+29427\sqrt{512}$
5	$886731088897+19594173939\sqrt{2048}$

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J.H.E. Cohn has remarked in [2, p. 21] that for the sequence l_n, there exist positive constants A and B such that

$$\frac{A2^n}{n} < l_n < B2^n n,$$

so that $\frac{\log l_n}{n} \to \log 2$ as $n \to \infty$.

Denoting the i-th convergent by A_i/B_i, the right hand inequality can be improved by using Cohn's inequality $B_{m-1} \geq F_m = (\frac{1+\sqrt{5}}{2})^m$ with $B_{m-1} = v_{n-1}$ from equation (2) below. For $u_{n-1} > \sqrt{D_{n-1}}v_{n-1}$ and hence

$$\begin{aligned} 2\sqrt{2^{2n-1}}v_{n-1} &< u_{n-1} + \sqrt{D_{n-1}}v_{n-1} = \epsilon_{n-1} = (1+\sqrt{2})^{2^{n-1}} \\ \left(\frac{1+\sqrt{5}}{2}\right)^m &< v_{n-1} < (1+\sqrt{2})^{2^{n-1}}/\sqrt{2^{2n+1}} \\ m &< \frac{2^{n-1}\log(1+\sqrt{2}) - \frac{(2n+1)}{2}\log 2}{\log\frac{1+\sqrt{5}}{2}} \\ l_n = 2m &< \frac{2^n\log(1+\sqrt{2}) - (2n+1)\log 2}{\log\frac{1+\sqrt{5}}{2}}. \end{aligned}$$

On the limited evidence from the table, perhaps $l_n/2^n \to .7427\cdots$.
Let $\sqrt{D_n} = [a_0, \overline{a_1, \ldots, a_{m-1}, a_m, a_{m+1}, \ldots, a_{2m}}]$, where $m = l_n/2$.
Lemma 3. The central partial quotient a_m is odd. More generally, if the length l of the period of the continued fraction of $\sqrt{D}$ is even, say $l = 2m$ and the fundamental solution $x_0 + y_0\sqrt{D}$ has y_0 odd, then a_m is odd.
Proof. Take $u = x_0, v = y_0, r = l_n = 2m$ in Lemma 1. Then because of the palindromic nature of $a_1, ..., a_{2m-1}$ (see [4, p. 81]), we have

$$
\begin{aligned}
\begin{pmatrix} Dy_0 & x_0 \\ x_0 & y_0 \end{pmatrix} &= A \begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix} A^t \\
&= \begin{pmatrix} x & y \\ a & b \end{pmatrix} \begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & a \\ y & b \end{pmatrix} \\
&= \begin{pmatrix} a_m x^2 + 2xy & a_m xa + ay + xb \\ a_m xa + ay + xb & a(a_m a + 2b) \end{pmatrix}.
\end{aligned}
$$

Hence $y_0 = a(a_m a + 2b)$ and so $a, a_m a + 2b$ and hence a_m, are odd.

Lemma 4. Let $(P_i + \sqrt{D})/Q_i$ denote the i-th complete convergent to $\sqrt{D_n}$. Then

$$A_{m-1} = 2u_{n-1}, B_{m-1} = v_{n-1}, m \text{ is even and } Q_m = 4, \text{ if } n > 1. \tag{2}$$

Proof. The statement is a consequence of Theorem 5, [3, p. 21]. However we will give a different proof. We have

$$
\begin{aligned}
u_{n-1}^2 - 2^{2n-1} v_{n-1}^2 &= 1 \\
(2u_{n-1})^2 - 2^{2n+1} v_{n-1}^2 &= 4.
\end{aligned}
$$

Then as $4 < \sqrt{D_n}$ if $n > 1$, it follows that $2u_{n-1}/v_{n-1} = A_{r-1}/B_{r-1}$ for some $r \geq 1$. Hence $A_{r-1} = 2u_{n-1}$ and $B_{r-1} = v_{n-1}$. Also $A_{r-1}^2 - D_n B_{r-1}^2 = (-1)^r Q_r$, so r is even and $Q_r = 4$.

Next we show that $r = m$. This will follow from the uniqueness result Lemma 5 below and the symmetry of the Q_i in the range $0 \leq i \leq m-1$ (see [4, p. 81]):

Lemma 5. If $Q_t = 4$ and $1 \leq t < 2m - 1$, then $t = r$.

Proof. $Q_t = 4$ implies $A_{t-1}^2 - D_n B_{t-1}^2 = (-1)^t 4$ and hence t is even. Also A_{t-1} is even. Hence

$$(A_{t-1}/2)^2 - D_{n-1} B_{t-1}^2 = 1$$

and

$$A_{t-1} + B_{t-1}\sqrt{D_{n-1}} = (u_{n-1} + v_{n-1}\sqrt{D_{n-1}})^i,$$

for some $i \geq 1$. But if $i \geq 2$, we would have the contradiction

$$v_n = B_{2m-1} > B_{t-1} \geq 2u_{n-1}v_{n-1} = 2v_n.$$

Hence $i = 1, B_{t-1} = v_{n-1} = B_{r-1}$, so $t = r$.
Lemma 6. $a_m = \lfloor\sqrt{D_{n-1}}\rfloor$ or $\lfloor\sqrt{D_{n-1}}\rfloor - 1$, whichever is odd.
Proof. $(P_m + \sqrt{D_n})/Q_m$ is reduced, so

$$-1 < (P_m - \sqrt{D_n})/Q_m < 0$$
$$\sqrt{D_n} - 4 < P_m < \sqrt{D_n}.$$

The symmetry of the P_i in the range $1 \le i \le m$ (see [4, p. 81]) then gives $P_m = P_{m+1}$. But $P_{m+1} = Q_m a_m - P_m = 4a_m - P_m$, so $P_m = 2a_m$. Hence

$$\sqrt{D_{n-1}} - 2 < a_m < \sqrt{D_{n-1}}$$

and $a_m = \lfloor\sqrt{D_{n-1}}\rfloor$ or $\lfloor\sqrt{D_{n-1}}\rfloor - 1$.
Examples.

1. $n = 2$. Here $l_n = 2$, $m = 1$, Also $D_{n-1} = 8$ and $\lfloor\sqrt{8}\rfloor = 2$. Hence $a_1 = \lfloor\sqrt{8}\rfloor - 1 = 1$.

2. $n = 4$. Here $l_n = 12$, $m = 6$, Also $D_{n-1} = 128$ and $\lfloor\sqrt{128}\rfloor = 11$. Hence $a_6 = \lfloor\sqrt{8}\rfloor = 11$.

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