A survey on Flow Polynomial

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Abstract

This expository article is about nowhere-zero flows and the flow polynomial, which counts the number of nowhere-zero flows of a graph. Following the definitions and properties of the flow polynomial, some examples and calculations are used to illustrate and develope the arithmetic of the flow polynomial. Furthermore, the flow polynomial of some classes of graphs are computed.

1 Introduction

Much information about the flow polynomial can be found in [4], [7] and [8]. Given a graph $G(V, E)$ with vertex set V and edge set E, where multiple edges are allowed, let (D, f) be an ordered pair where D is an orientation of $E(G)$ and $f : E(G) \to \mathbf{Z}$ be an integer-valued function called a *flow*. An oriented edge of G is called an *arc*. For a vertex $v \in V(G)$, let $E^+(v) =$ {all arcs of D(G) with their tails at v} and $E^-(v) =$ {all arcs of D(G) with their heads at v}.

Definition 1.1 *A* **λ-flow** *of a graph G is a flow f such that $|f(e)| < \lambda$ for every edge $e \in E(G)$ and for every vertex $v \in V(G)$*

$$\sum_{e\in E^+(v)} f(e) \equiv \sum_{e\in E^-(v)} f(e) \quad (mod \quad \lambda).$$

The **support** of f, $supp(f)$, is the set of all edges of G with $f(e) \neq 0$. A λ-flow is **nowhere-zero** if $supp(f) = E(G)$.

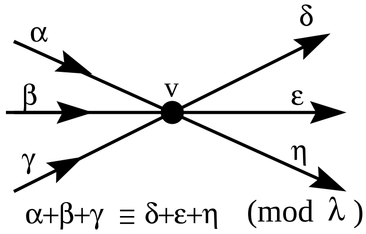


Figure 1: A λ-flow at vertex V

Some long-standing conjectures on flows found in [9] are as follows:

Conjecture 1.2 *Every bridgeless graph admits a nowhere-zero 5-flow.*

Conjecture 1.3 *Every bridgeless graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow.*

Conjecture 1.4 *Every bridgeless graph containing no 3-edge-cut admits a nowhere-zero 3-flow.*

A number $\Lambda(G)$ of interest is the least integer Λ such that G has a nowhere-zero Λ-flow. In [2], Jaeger increased the plausibility of conjecture 1.2 by proving that every bridgeless graph has a nowhere-zero 8-flow. Seymour [5], improved this upper bound Λ to 6. In [6], $\Lambda(G)$ was further lowered for certain classes of graphs. For a graph $G(V, E)$, the **cyclomatic number** of G, $\nu(G)$ is defined as $\nu(G) = |E(G)| - |V(G)| + \kappa(G)$ where $\kappa(G)$ denotes the number of components. In [8], Tutte defines the **flow polynomial**, $F(G, \lambda)$, of a graph G as a graph function and as a polynomial in an indeterminate λ with integer coefficients by

$$F(G, \lambda) = (-1)^{|E(G)|} \sum_{S \subseteq E(G)} (-1)^{|S|} \lambda^{\nu(G:S)}$$

where $(G : S)$ denotes the spanning subgraph of G with edge-set S. $F(G, \lambda)$ is a polynomial in λ which gives the number of nowhere-zero λ-flows in G independent of the chosen orientation. Tutte [8] defines the **chromatic polynomial**, $P(G, \lambda)$, of a graph G by

$$P(G, \lambda) = \sum_{S \subseteq E(G)} (-1)^{|S|} \lambda^{\kappa(G:S)}.$$

When λ takes a positive integral value n, $P(G, n)$ is the number of "proper vertex" n-colorings of G. For more information on chromatic polynomials see [3]. It is often more convenient to work with the new variable $\omega = 1 - \lambda$. Tutte [8] states seven properties of the flow polynomial $F(G)$ of a graph G, where G can be any graph, possibly with multiple edges and/or loops, as follows:

Property 1.5 *$F(G,\omega)$ is a polynomial of degree $\nu = \nu(G)$. Coefficient of ω^ν is $(-1)^\nu$ and all terms in $F(G,\omega)$ have the same sign.*

Property 1.6 *If G has no edges, then $F(G,\lambda) = 1$.*

Property 1.7 *If G has a bridge, then $F(G,\lambda) = 0$.*

Property 1.8 *If G consists of two graphs H and K which are either disjoint or have a single vertex in common, then $F(G,\lambda) = F(H,\lambda) \cdot F(K,\lambda)$.*

Property 1.9 *If G is a cycle, then $F(G,\lambda) = \lambda - 1$.*

Property 1.10 *If e is any edge of G, then $F(G,\lambda) = F(G'',\lambda) - F(G',\lambda)$, where G' and G'' are obtained from G by deleting and contracting the edge e, respectively.*

Property 1.11 *$F(G,\lambda)$ is a topological invariant and hence any two homeomorphic graphs will have the same flow polynomial.*

By a result of Jaeger [1], if G is planar, then $P(G^*,\lambda) = \lambda \cdot F(G,\lambda)$, where G^* is the planar dual of G.

2 Some Examples and Calculations

To illustrate the properties discussed above, we compute the flow polynomial of some graphs.

Example 2.1 *Given G, H and K in Figure 2, $F(G,\lambda) = F(H,\lambda) = F(K,\lambda)$.*

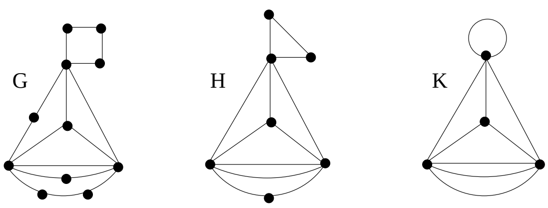


Figure 2: Homeomorphic graphs & suppression of degree 2 vertices

Example 2.2 *Given G, G^*, H and H^* in Figure 3, we have*

$$F(G,\lambda) = \frac{1}{\lambda}P(G^*,\lambda) = \frac{1}{\lambda}\left[\lambda(\lambda-1)(\lambda-2)\right] = (\lambda-1)(\lambda-2)$$

$$F(H,\lambda) = \frac{1}{\lambda}P(H^*,\lambda) = \frac{1}{\lambda}\left[\lambda(\lambda-1)^3\right] = (\lambda-1)^3$$

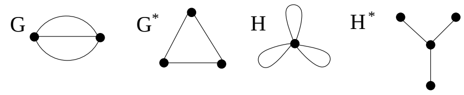


Figure 3: Some planar graphs with their planar duals

Example 2.3 *Let X_3 denote the 2-connected graph on 2 vertices with 3 edges. As we just saw, $F(X_3, \lambda) = \lambda^2 - 3\lambda + 2$ and the number of all nowhere-zero 4-flows of X_3 is $F(X_3, 4) = 6$. In Figure 4 we list all of them for the arbitrary orientation that we have picked for X_3*

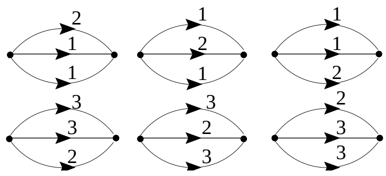


Figure 4: The 6 nowhere-zero 4-flows of X_3

Example 2.4 *Let X_5 denote the 2-connected graph on 2 vertices with* 5 *edges. By Lemma 4.1, $F(X_5, \omega) = \omega + \omega^2 + \omega^3 + \omega^4$. Hence the number of all nowhere-zero 3-flows of X_5 is $F(X_5, -2) = 10$. In Figure 5 we list all of them for the arbitrary orientation that we have picked for X_5*

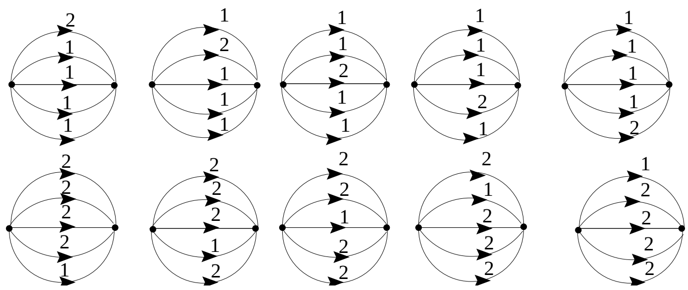
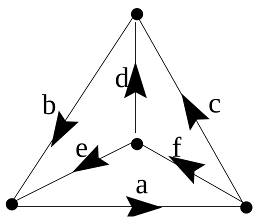


Figure 5: The 10 nowhere-zero 3-flows of X_5

Example 2.5 *Let K_4 be the complete graph on 4 vertices with flows as shown in Figure 6. Since $F(K_4, \lambda) = \frac{1}{\lambda} P(K_4^*, \lambda) = \frac{1}{\lambda} P(K_4, \lambda) = -6 + 11\lambda - 6\lambda^2 + \lambda^3$, the the number of nowhere-zero 4-flows of K_4 is $F(K_4, 4) = 6$. In Figure 6 we list all of them for the arbitrary orientation that we have picked for K_4*



a	b	c	d	e	f
3	2	1	1	1	2
2	3	1	2	-1	1
-2	-1	-3	2	-1	1
-1	1	-2	3	-2	1
1	-2	-1	-1	3	2
2	1	-1	2	1	3

Figure 6: The graph K_4 and its 6 nowhere-zero 4-flows

3 Sheaf Removal and Duality

Given a graph M consider a bundle of multiplicity n and let K be the graph obtained by contracting this bundle in G to a vertex and H that obtained by deleting this bundle. By using Property 1.10 of flow polynomials repeatedly, Read and Whitehead [4] arrive at the "SRF", or the **Sheaf Removal Formula**:

$$F(M,\omega) = (-1)^n \left[\frac{\omega^n - 1}{1-\omega} F(K,\omega) + F(H,\omega) \right]. \tag{3.1}$$

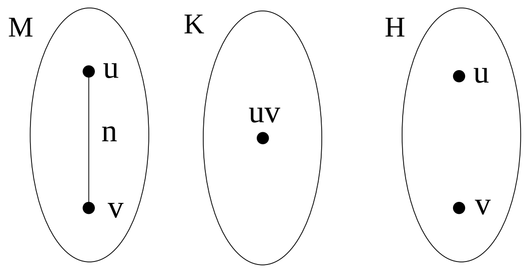


Figure 7: The Deletion and Contraction of a sheaf of edges

The chromatic polynomial and the flow polynomial are in general related to each other by the following formulas of Read and Whitehead in [4]:

$$F(M,\lambda) = \frac{(-1)^{\mu(E)}}{\lambda^p} \sum_{U \subset E} P(M_U)(1-\lambda)^{\mu(U)} \tag{3.2}$$

$$P(G,\lambda) = \frac{(-1)^{\mu(E)}}{\lambda^{q-p}} \sum_{(Y,U)} F(Y)(1-\lambda)^{\mu(U)} \tag{3.3}$$

In Equation 3.2, $\mu(U)$ is the sum of the number of edges in the bundles of U, while in Equation 3.3, $\mu(U)$ denotes the sum of lengths of the chains in U. Equation 3.2 will be explained later. Here we show how Equation 3.3 works by letting $G = K_4$. We find all the spanning subgraphs Y of K_4 and list the number of different ones in each class.

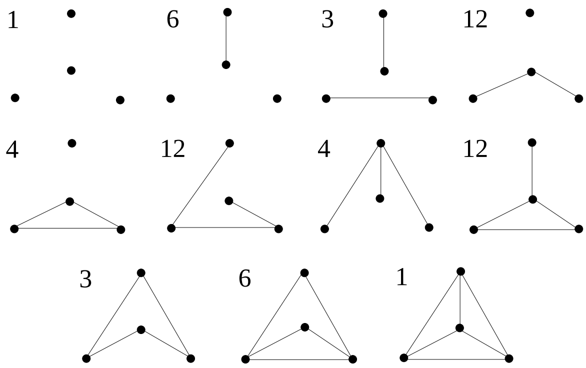


Figure 8: Y, Spanning subgraphs K_4

Next for each Y, we find its complement U.

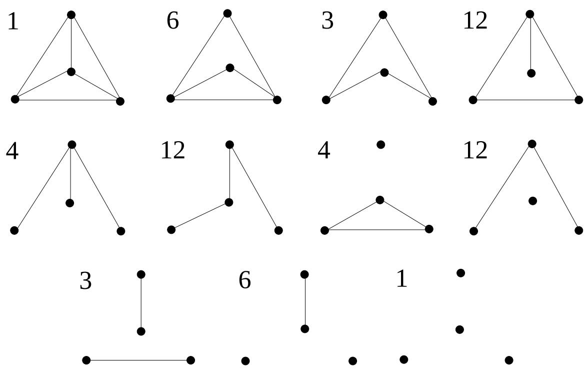


Figure 9: U, Spanning complements of Y

Using Equation 3.3 and Property 1.7, we obtain

$$\begin{aligned}
P(K_4, \omega) &= \frac{(-1)^6}{\lambda^2}\Big[1 \cdot (\omega^6) + 0 + 0 + 0 - 4\omega(\omega^3) + 0 + 0 + 0 \\
&- 3\omega(\omega^2) + 6(\omega + \omega^2) \cdot (\omega^1) + (-2\omega - 3\omega^2 - \omega^3) \cdot (\omega^0)\Big] \\
&= \frac{(-1)^6}{\lambda^2}(\omega^6 - 4\omega^4 + 2\omega^3 + 3\omega^2 - 2\omega) = \omega^4 + 2\omega^3 - \omega^2 - 2\omega \\
P(K_4, \omega) &= (1 - \omega)(-\omega)(-1 - \omega)(-2 - \omega) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)
\end{aligned}$$

A graph G with loops has no proper vertex coloring, i.e, $P(G, \lambda) = 0$. Likewise, we have already seen that a graph M with bridges can not have any nowhere-zero λ-flows, i.e., $F(M, \lambda)$=0.

Two graphs are *homeomorphic* if both can be obtained from the same graph by inserting new vertices of degree 2 into its edges. Graphs with the same underlying simple graph were given the name *amallamorphs* by Read and Whitehead in [4]. Two graphs G and H are said to be *chromatically equivalent* if $P(G, \lambda) = P(H, \lambda)$, while two graphs G and H are said to be *flow equivalent* if $F(G, \lambda) = F(H, \lambda)$.

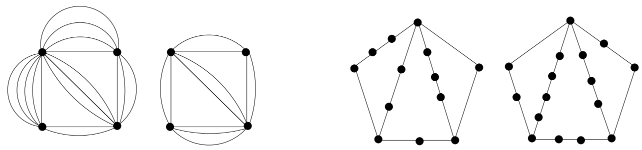


Figure 10: Amallamorphic graphs and homeomorphic graphs

Since multiple edges have no effect on the colorings of the vertices involved, all amallamorphic graphs have the same chromatic polynomial. Similarly by Property 1.11, all degree 2 vertices can be suppressed and that is why all homeomorphic graphs have the same flow polynomial. We summarize the above in the Table 11. The decomposition property for colorings holds, if G consists of two graphs H and K which are disjoint.

Property	*Vertex λ-Colorings*	*Nowhere-Zero λ-Flows*
Polynomial	chromatic $P(G)$	flow $F(G)$
Degree	$\lvert V \rvert$	$\lvert E \rvert - \lvert V \rvert + \kappa(G)$
Unity	no vertices $P(G) = 1$	no edges $F(G) = 1$
Reduction	$P(G) = P(G - e) - P(G_e)$	$F(G) = F(G_e) - F(G - e)$
Decomposition	$P(G) = P(H) \cdot P(K)$	$F(G) = F(H) \cdot F(K)$
Annihilation	with loops, $P(G) = 0$	with bridges, $F(G) = 0$
Equivalence	amallamorphism	homeomorphism
Main Result	4-Color Theorem	Nowhere-Zero 6-Flow
Connection	*Jaeger*: If G is planar,	then $P(G) = \lambda F(G^*)$

Table 11: The comparison of colorings and flows

4 The Fundamental Elements

Let X_n denote the 2-connected graph on 2 vertices with n edges and L_n denote the graph with n loops and one vertex. One might call these the *fundamental elements* from which other flow polynomials are computed.

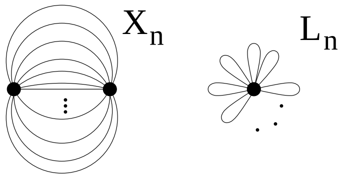


Figure 12: The graphs X_n and L_n

Lemma 4.1 $F(L_n, \omega) = (-\omega)^n, F(X_n, \omega) = (-1)^{n+1} \sum_{i=1}^{n-1} \omega^i$ *for* $n \geq 2$.

Proof: By Property 1.9 we know that the flow polynomial of the cycle is $-\omega$. By Property 1.8 $F(L_n, \omega) = \underbrace{(-\omega)(-\omega)\ldots(-\omega)}_{n} = (-\omega)^n$. As for X_n, use induction on the number of edges in X_n. For $n = 2$, $F(X_2, \omega) = -\omega$. Suppose $F(X_n, \omega) = (-1)^{n+1} \sum_{i=1}^{n-1} \omega^i$. Take X_{n+1} and apply Property 1.10 to any edge e. Then

$$F(X_{n+1}, \omega) = -F(X_n) + F(L_n) = -(-1)^{n+1} \sum_{i=1}^{n-1} \omega^i + (-\omega)^n$$

$$= \quad (-1)^{n+2} \sum_{i=1}^{n-1} \omega^i + (-1)^{n+2} (\omega)^n = (-1)^{n+2} \sum_{i=1}^{n} \omega^i \qquad \blacksquare$$

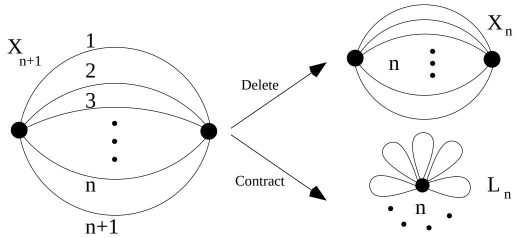


Figure 13: Applying the Deletion-Contraction Principle to X_{n+1}

5 Graphs With Prescribed Multiplicities

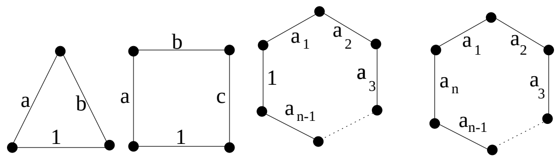


Figure 14: $M_3(a, b, 1)$, $M_4(a, b, c, 1)$, $M_n(a_1, \ldots, a_{n-1}, 1)$ and $M_n(a_1, \ldots, a_n)$

We now focus our attention on $M_n(a_1, a_2, \ldots, a_n)$, whose underlying simple graphs are the circuits, C_n.

Theorem 5.1 *Let C_n be the underlying simple graph of the graph M_n with edge multiplicities $a_1, a_2, \ldots, a_{n-1}, 1$. Then*

$$F(M_n, \omega) = (-1)^{a_1+a_2+\ldots+a_{n-1}+2-n}\omega \prod_{j=1}^{n-1} \sum_{i=0}^{a_j-1} \omega^i.$$

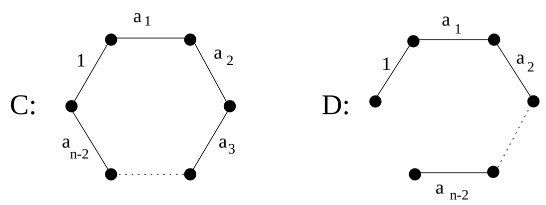


Figure 15: Applying SRF to M_n

Proof: We use induction on n and apply the SRF to the bundle whose edge multiplicity is a_{n-1}. Contraction and deletion of this edge bundle yields the graphs C and D shown in Figure 15.

$$F(M_n, \omega) = (-1)^{a_{n-1}} \left[\frac{\omega^{a_{n-1}} - 1}{1 - \omega} F(C) + F(D) \right]$$

$$= \quad (-1)^{a_{n-1}} \left[-(1 + \omega + \ldots + \omega^{a_{n-1}-1}) F(C) + 0 \right] = (-1)^{a_{n-1}-1}$$

$$(1 + \omega + \ldots + \omega^{a^{n-1}-1})(-1)^{a_1+a_2+\cdots+a_{n-2}+2-n+1}\omega \prod_{j=1}^{n-2} \sum_{i=0}^{a_j-1} \omega^i$$

$$= \quad (-1)^{a_1+a_2+\ldots+a_{n-1}+2-n}\omega \prod_{j=1}^{n-1} \sum_{i=0}^{a_j-1} \omega^i$$

In the above, D had a bridge. Therefore, by Property 1.7 $F(D) = 0$. ■

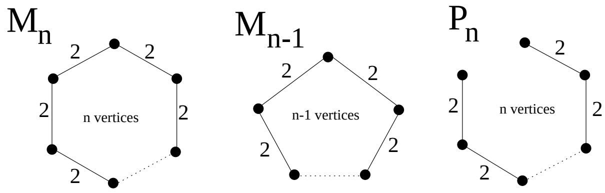


Figure 16: M_n, M_{n-1} and P_n with all edge multiplicities 2

Theorem 5.2 *Let C_n be the underlying simple graph of the graph M_n with all edge multiplicities 2. For $n \geq 2$, $F(M_n, \omega) = (-1)^{n+1}[\omega(1+\omega)^n - \omega^n]$.*

Proof: We proceed by induction. For $n = 2$, $F(M_2, \omega) = F(X_4, \omega) = -\omega - \omega^2 - \omega^3 = (-1)^{2+1}[\omega(1+\omega)^2 - \omega^2]$. Suppose that $F(M_{n-1}, \omega) = (-1)^n[\omega(1+\omega)^{n-1} - \omega^{n-1}]$. We apply the SRF to any bundle of edge multiplicity 2. Contraction and deletion of this edge bundle yields the graphs M_{n-1} and P_n shown in Figure 16. Hence we have

$$\begin{aligned} F(M_n, \omega) &= (-1)^2 \left[\frac{\omega^2 - 1}{1 - \omega} F(M_{n-1}, \omega) + F(P_n, \omega) \right] \\ &= -(1+\omega)(-1)^n \left[\omega(1+\omega)^{n-1} + (-1)^{n-1}\omega^{n-1} \right] + (-\omega)^{n-1} \\ &= (-1)^{n+1}\omega(1+\omega)^n + (-1)^n\omega^{n-1} + (-1)^n\omega^n \\ &+ (-1)^{n-1}\omega^{n-1} = (-1)^{n+1}\omega(1+\omega)^n + (-1)^n\omega^n \quad \blacksquare \end{aligned}$$

Now we try to find the flow polynomial of the general cycle graph $M_n(a_1, a_2, \ldots, a_n)$ depicted in Figure 17.

Theorem 5.3 *Assume $M_n(a_1, a_2, \ldots, a_n)$ has C_n as its underlying simple graph with edge multiplicities $a_1, a_2, \ldots, a_n$. Then for $n \geq 3$*

$$F(M_n, \omega) = (-1)^{\sum_{i=1}^{n-2}(a_{n+1-i})} \cdot \frac{F(M_2, \omega)}{(1-\omega)^{n-2}} \cdot \prod_{j=1}^{n-2}(\omega^{a_{n+1-j}} - 1)$$
$$+ \sum_{j=1}^{n-2} \left((-1)^{\sum_{i=1}^{n-1-j}(a_{n+1-i})} \cdot F(P_{j+2}, \omega) \cdot \prod_{m=1}^{n-2-j} \left(\frac{\omega^{a_{n+1-m}} - 1}{(1-\omega)^{n-2-j}} \right) \right)$$

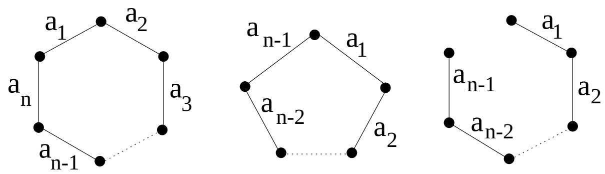


Figure 17: $M_n(a_1, \ldots, a_n)$, $M_{n-1}(a_1, \ldots, a_{n-1})$ and $P_n(a_1, \ldots, a_{n-1})$

Proof: Here we let $M_i = M_i(a_1, \ldots, a_i)$, while $P_i = P_i(a_1, \ldots, a_{i-1})$ and $M_2 = M_2(a_1, a_2)$ are the graphs shown in Figure 18. Also $M_2 \cong X_{a_1+a_2}$. We proceed by induction. For $n = 3$,

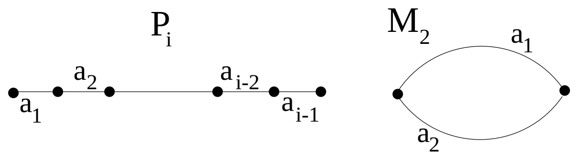


Figure 18: The graphs P_i and M_2

$$F(M_3,\omega) = (-1)^{a_3}\frac{F(M_2,\omega)}{1-\omega}\cdot(\omega^{a_3}-1) + (-1)^{a_3}F(P_3,\omega).$$

However, the above is merely an application of the deletion-contraction principle for the edge bundle a_3 of M_3. Now suppose that $F(M_k,\omega)$ is known. We apply the SRF to the bundle of edge multiplicity a_{k+1} of M_{k+1}. Contraction and deletion of this edge bundle yields the graphs M_k and P_{k+1}. Hence we have

$$\begin{aligned}
& F(M_{k+1},\omega) = (-1)^{a_{k+1}}\left\{\frac{\omega^{a_{k+1}}-1}{1-\omega}F(M_k,\omega) + F(P_{k+1},\omega)\right\} \\
= \;& (-1)^{a_{k+1}}\left\{\frac{\omega^{a_{k+1}}-1}{1-\omega}\left[(-1)^{\sum_{i=1}^{k-2}(a_{k+1-i})}\cdot\frac{F(M_2,\omega)}{(1-\omega)^{k-2}}\cdot\prod_{j=1}^{k-2}(\omega^{a_{k+1-j}}-1)\right.\right. \\
+ \;& \left.\sum_{j=1}^{k-2}\left((-1)^{\sum_{i=1}^{k-1-j}(a_{k+1-i})}\cdot F(P_{j+2},\omega)\cdot\prod_{m=1}^{k-2-j}\left(\frac{\omega^{a_{k+1-m}}-1}{(1-\omega)^{k-2-j}}\right)\right)\right] \\
+ \;& \left. F(P_{k+1},\omega)\right\} = (-1)^{\sum_{i=0}^{k-2}(a_{k+1-i})}\cdot\frac{F(M_2,\omega)}{(1-\omega)^{k-1}}\cdot\prod_{j=0}^{k-2}(\omega^{a_{k+1-j}}-1) \\
+ \;& \sum_{j=1}^{k-1}\left((-1)^{\sum_{i=0}^{k-1-j}(a_{k+1-i})}\cdot F(P_{j+2},\omega)\cdot\prod_{m=1}^{k-1-j}\left(\frac{\omega^{a_{k+2-m}}-1}{(1-\omega)^{k-1-j}}\right)\right)
\end{aligned}$$

However, at his point a simple shift in all the indices will change the last statement to the following.

$$\begin{aligned}
& F(M_{k+1},\omega) = (-1)^{\sum_{i=1}^{k-1}(a_{k+2-i})}\cdot\frac{F(M_2,\omega)}{(1-\omega)^{k-1}}\cdot\prod_{j=1}^{k-1}(\omega^{a_{k+2-j}}-1) \\
+ \;& \sum_{j=1}^{k-1}\left((-1)^{\sum_{i=1}^{k-j}(a_{k+2-i})}\cdot F(P_{j+2},\omega)\cdot\prod_{m=1}^{k-1-j}\left(\frac{\omega^{a_{k+2-m}}-1}{(1-\omega)^{k-1-j}}\right)\right)
\end{aligned}$$

And this is exactly what we wanted to show. ∎

We now express the flow polynomial of M_3 not as a rational function. Table 20 provided some insight as to what the formula should be.

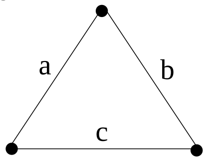


Figure 19: The graph $M_3(a, b, c)$

a	**b**	**c**	ν	coefficients of $F(M_3, \omega)$ in ascending powers of ω
2	3	8	11	-1,-3,-4,-4,-4,-4,-4,-4,-3,-2,-1
6	6	6	16	1,3,5,7,9,11,10,9,8,7,6,5,4,3,2,1
3	8	8	17	-1,-3,-5,-6,-7,-8,-9,-10,-9,-8,-7,-6,-5,-4,-3,-2,-1
4	7	9	18	1,3,5,7,8,9,10,10,10,9,8,7,6,5,4,3,2,1

Table 20: Flow polynomial of $M_3(a, b, c)$ for selected edge multiplicities

Notice that the coefficients, in absolute value, start at 1 and increase through the odd numbers, then go up by consecutive integers, reach a plateau and stay there for a while and then decrease back to 1.

Theorem 5.4 *Let K_3 be the underlying simple graph of the graph M_3 with edge multiplicities a, b, c where $0 < a \leq b \leq c$, and $a, b, c \in \mathbf{N}$. Then*

$$F(M_3, \omega) = (-1)^{a+b+c}\left[\sum_{i=1}^{a}(2i-1)\omega^i + \sum_{i=a+1}^{b}(a-1+i)\omega^i + \sum_{i=b+1}^{c}(a+b-1)\omega^i + \sum_{i=c+1}^{a+b+c-2}(a+b+c-1-i)\omega^i\right].$$

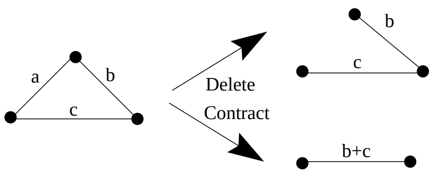


Figure 21: SRF applied to the bundle a

Proof: We apply the SRF to the bundle whose edge multiplicity is a. Contraction and deletion of edge bundle a is depicted in Figure 21.

$$F(M_3, \omega) = (-1)^a\left[\frac{\omega^a - 1}{1-\omega}F(X_{b+c}, \omega) + F(X_b, \omega) \cdot F(X_c, \omega)\right]$$

$$
\begin{aligned}
&= (-1)^a \left[\frac{\omega^a - 1}{1-\omega} (-1)^{b+c-1} (\omega + \omega^2 + \ldots + \omega^{b+c-1}) \right. \\
&+ \left. (-1)^{b-1} (\omega + \ldots + \omega^{b-1}) (-1)^{c-1} (\omega + \ldots + \omega^{c-1}) \right] \\
&= (-1)^{a+b+c} \left[\underbrace{(\omega + \ldots + \omega^{a-1})(\omega + \ldots + \omega^{b+c-1})}_{P} \right. \\
&+ \left. \underbrace{(\omega + \omega^2 + \ldots + \omega^{b-1})(\omega + \omega^2 + \ldots + \omega^{c-1})}_{Q} \right]
\end{aligned}
$$

Powers of ω	1	2	a	b	c	b+c-2	b+c-1	a+b+c-2
Coeff of P	1	2 ↗	a →	a →	a→	a →	a ↘	1
Coeff of Q	0	1 ↗	a-1 ↗	b-1 →	b-1 ↘	1	0	0
Sum of coeff	1	3 ↗	2a-1 ↗	a+b-1 →	a+b-1 ↘	a+1	a ↘	1

Table 22: Collection of coefficients of ω^i

Upon multiplying out and collecting the terms in the products P and Q, we see a number of breaks in the ascending powers of ω where these powers can be linearly ordered. We gather the similar terms and add the coefficients of ω^i in Table 22. Upon adding all the terms, the result follows. ■

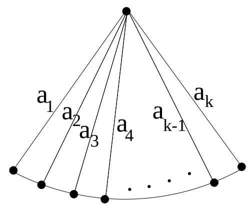


Figure 23: The sector graph

In [7], it was shown that the flow polynomial of the sector graph $S_k = S_k(a_1, a_2, \ldots, a_n)$, shown in Figure 23, is

$$F(S_k, \omega) = (-1)^{(k-1+\sum_{i=1}^{k} a_i)} \frac{(\omega^{a_1} - 1)(\omega^{a_k} - 1)}{(1-\omega)^2} \left(\prod_{i=2}^{k-1} \frac{\omega^{1+a_i} - 1}{1-\omega} \right).$$

Theorem 5.5 *Given $k \geq 2$ and $W_k(a_1, a_2, \ldots, a_k)$ whose underlying simple graph is W_k, the wheel with k spokes, we have*

$$F(W_k, \omega) = (-1)^{k+\sum_{i=1}^{k} a_i} \cdot \omega \cdot \left[\sum_{i=1}^{k-1} (-1)^{i+1} \right.$$

$$\frac{\left(\omega^{a_{k+1-i}}-1\right)\prod_{j=1}^{k-i}\left(\omega^{1+a_{k+1-i-j}}-1\right)}{(1-\omega)^{k+1-i}}+(-1)^{k+1}\frac{\omega^{a_1}-\omega}{1-\omega}\Bigg].$$

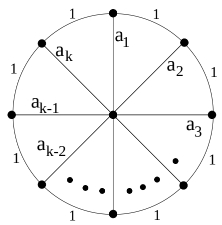


Figure 24: The wheel $W_k(a_1, a_2, \ldots, a_k)$

Proof: We proceed by induction. For $k = 2$, the formula gives

$$F(W_2,\omega)=(-1)^{2+a_1+a_2}\cdot\omega\left[(-1)^2\frac{(\omega^{a_2}-1)(\omega^{1+a_1}-1)}{(1-\omega)^2}-\frac{\omega^{a_1}-\omega}{1-\omega}\right]$$

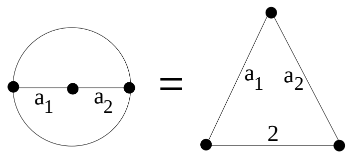


Figure 25: The wheel W_2 with a redrawing of it

To verify this, we start with W_2 and apply SRF to some edge bundle, say a_2. Then we obtain $F(W_2, \omega)$=

$$\begin{aligned}
&= (-1)^{a_2}\left[\frac{\omega^{a_2}-1}{1-\omega}F(X_{2+a_1},\omega)+F(X_{a_1},\omega)F(X_2,\omega)\right]\\
&= (-1)^{a_2}\left[\frac{\omega^{a_2}-1}{1-\omega}\cdot(-1)^{1+a_1}\frac{\omega^{2+a_1}-\omega}{1-\omega}+(-1)^{1+a_1}\omega\frac{\omega^{a_1}-\omega}{1-\omega}\cdot(-\omega)\right]\\
&= (-1)^{1+a_1+a_2}\frac{\omega^{a_2}-1}{1-\omega}\cdot\frac{-\omega(\omega^{1+a_1}-1)}{1-\omega}+(-1)^{1+a_1+a_2}\omega\frac{\omega^{a_1}-\omega}{1-\omega}\\
&= (-1)^{2+a_1+a_2}\cdot\omega\left[(-1)^2\frac{(\omega^{a_2}-1)(\omega^{1+a_1}-1)}{(1-\omega)^2}-\frac{\omega^{a_1}-\omega}{1-\omega}\right]
\end{aligned}$$

Now suppose the result is true for $k = n$. Start with W_{n+1} and apply SRF to the bundle whose edge multiplicity is a_{n+1}. The result is shown in Figure 26.

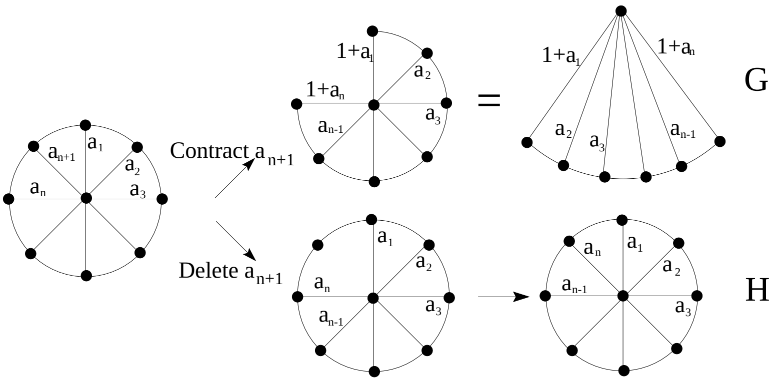


Figure 26: Decomposition results in sector and wheel graphs

$$F(W_{n+1},\omega) \quad = \quad (-1)^{a_{n+1}}\left[\frac{\omega^{a_{n+1}}-1}{1-\omega}F(G,\omega)+F(H,\omega)\right]$$

In Figure 26, the graph obtained from W_{n+1} by deletion of bundle a_{n+1} is homeomorphic to H by Property 1.11 where $H \cong W_n$, while the one obtained from contracting a_{n+1} is S_n, a sector graph whose flow polynomial is known. Hence we now have

$$\begin{aligned}
& F(W_{n+1},\omega) = (-1)^{a_{n+1}}\left[\frac{\omega^{a_{n+1}}-1}{1-\omega}\cdot(-1)^{n-1+\sum_{i=1}^{n}a_i+2}\cdot\right. \\
& \left.\omega\frac{\omega^{1+a_1}-1}{1-\omega}\cdot\frac{\omega^{1+a_n}-1}{1-\omega}\cdot\prod_{i=2}^{n-1}\left(\frac{\omega^{1+a_i}-1}{1-\omega}\right)+F(W_{n-1},\omega)\right] \\
= \quad & (-1)^{n+1+\sum_{i=1}^{n+1}a_i}\cdot\omega\frac{\omega^{a_{n+1}}-1}{1-\omega}\cdot\prod_{i=1}^{n}\left(\frac{\omega^{1+a_i}-1}{1-\omega}\right) \\
+ \quad & (-1)^{a_{n+1}}F(W_n,\omega) = (-1)^{n+1+\sum_{i=1}^{n+1}a_i}\cdot\omega\frac{\omega^{a_{n+1}}-1}{1-\omega}\cdot \\
& \prod_{i=1}^{n}\left(\frac{\omega^{1+a_i}-1}{1-\omega}\right)+(-1)^{a_{n+1}}\left((-1)^{n+\sum_{i=1}^{n}a_i}\cdot\omega\left[\sum_{i=1}^{n-1}(-1)^{i+1}\right.\right. \\
& \left.\left.\frac{\left(\omega^{a_{n+1-i}}-1\right)\prod_{j=1}^{n-i}\left(\omega^{1+a_{n+1-i-j}}-1\right)}{(1-\omega)^{n+1-i}}+(-1)^{n+1}\frac{\omega^{a_1}-\omega}{1-\omega}\right]\right) \\
= \quad & (-1)^{n+1+\sum_{i=1}^{n+1}a_i}\cdot\omega\cdot\left\{\frac{\omega^{a_{n+1}}-1}{1-\omega}\cdot\prod_{i=1}^{n}\left(\frac{\omega^{1+a_i}-1}{1-\omega}\right)+\left[\sum_{i=1}^{n-1}\right.\right.
\end{aligned}$$

$$(-1)^{i+1}\frac{\left(\omega^{a_{n+1-i}}-1\right)\prod_{j=1}^{n-i}\left(\omega^{1+a_{n+1-i-j}}-1\right)}{(1-\omega)^{n+1-i}}+(-1)^{n+1}\frac{\omega^{a_1}-\omega}{1-\omega}\Bigg]\Bigg\}$$

$$= (-1)^{n+1+\sum_{i=1}^{n+1}a_i}\cdot\omega\cdot\Bigg\{\frac{\omega^{a_{n+1}}-1}{1-\omega}\cdot\prod_{i=1}^{n}\left(\frac{\omega^{1+a_i}-1}{1-\omega}\right)+\Bigg[\sum_{i=1}^{n-1}$$

$$(-1)^{i}\frac{\left(\omega^{a_{n+1-i}}-1\right)\prod_{j=1}^{n-i}\left(\omega^{1+a_{n+1-i-j}}-1\right)}{(1-\omega)^{n+1-i}}+(-1)^{n+1}\frac{\omega^{a_1}-\omega}{1-\omega}\Bigg]\Bigg\}$$

By closely studying $\left\{\ldots+\left[\ldots\right]\right\}$ in the last Equation, we can see that the first term can be absorbed by the second by lowering the index in the sum from 1 to 0.

$$= (-1)^{n+1+\sum_{i=1}^{n+1}a_i}\cdot\omega\cdot\Bigg\{\sum_{i=1}^{n}(-1)^{i}$$

$$\frac{\left(\omega^{a_{n+1-i}}-1\right)\prod_{j=1}^{n-i}\left(\omega^{1+a_{n+1-i-j}}-1\right)}{(1-\omega)^{n+1-i}}+(-1)^{n+1}\frac{\omega^{a_1}-\omega}{1-\omega}\Bigg\}$$

However now by a readjustment of the index of summation, we obtain

$$= (-1)^{n+1+\sum_{i=1}^{n+1}a_i}\cdot\omega\cdot\Bigg\{\sum_{i=0}^{n-1}(-1)^{i}$$

$$\frac{\left(\omega^{a_{n+1-i}}-1\right)\prod_{j=1}^{n-i}\left(\omega^{1+a_{n+1-i-j}}-1\right)}{(1-\omega)^{n+1-i}}+(-1)^{n+1}\frac{\omega^{a_1}-\omega}{1-\omega}\Bigg\}$$

which is the desired result and completes the inductive proof. ■

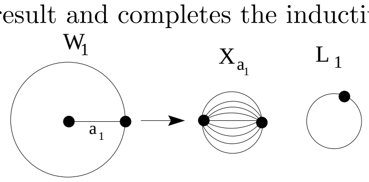


Figure 27: The wheel W_1

As the reader might have noticed by now, the above induction has an initial starting point at $n = 2$. The first wheel W_1 is a degenerate case and must be dealt with separately. This is however a very trivial case and as Figure 27 shows, W_1 can be factored as the disjoint union of the X_{a_1} and L_1 by Property 1.8. So

$$\begin{aligned} F(W_1,\omega) &= F(X_{a_1},\omega)\cdot F(X_2,\omega) \\ &= (-1)^{1+a_1}\frac{w^{a_1-1}-1}{\omega-1}\cdot(-\omega) = (-1)^{a_1}\frac{w^{a_1}-\omega}{\omega-1} \end{aligned}$$

6 Expansion On Certain Subgraphs

As we explained in Section 3, the flow polynomial of a graph M can be expressed as a polynomial in $\omega = 1-\lambda$, where the coefficients of ω^i are chromatic polynomials of certain subgraphs of M. See Equation 3.2.

Lemma 6.1 *Let C_3 be the underlying simple graph of the graph M_3 with edge multiplicities a, b, c. Then*

$$F(M_3,\omega) = \frac{(-1)^{a+b+c}}{(1-\omega)^3}\left[(\omega-\omega^3)+(\omega^2-\omega)(\omega^a+\omega^b+\omega^c)+(1-\omega)\omega^{a+b+c}\right].$$

Lemma 6.2 *Let C_4 be the underlying simple graph of the graph M_4 with edge multiplicities a, b, c, d. Then $F(M_4,\omega) =$*

$$\frac{(-1)^{a+b+c+d}}{(1-\omega)^4}\Bigg[(\omega^4-\omega)+(\omega-\omega^3)(\omega^a+\omega^b+\omega^c+\omega^d)+(\omega^2-\omega)\cdot$$
$$(\omega^{a+c}+\omega^{b+d}+\omega^{a+b}+\omega^{a+d}+\omega^{b+c}+\omega^{c+d})+(1-\omega)\omega^{a+b+c+d}\Bigg]$$

Proof: We look at all different classes of subgraphs of M_3 here and use 3.2. In the $< U >$ column of Table 28, we list representative subgraphs. In the G_U column, the complement of each representative subgraph, with all the edges in U contracted to point, are listed.

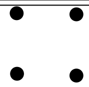
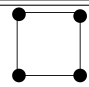
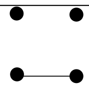
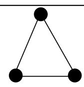
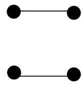
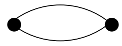
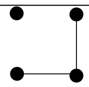
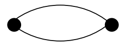
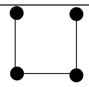
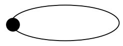
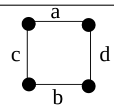
$< U >$	G_U	$P(G_U, \lambda)$	$P(G_U, \omega)$	Powers of ω
		$(\lambda - 1)^4 + (\lambda - 1)$	$-\omega + \omega^4$	0
		$\lambda(\lambda - 1)(\lambda - 2)$	$\omega - \omega^3$	a b c d
		$\lambda(\lambda - 1)$	$-\omega + \omega^2$	$a + b$ $c + d$
		$\lambda(\lambda - 1)$	$-\omega + \omega^2$	$a + c$ $a + d$ $b + d$ $b + c$
		0	0	$a + c + d$ $a + b + d$ $b + c + d$ $a + b + c$
		λ	$1 - \omega$	$a + b + c + d$

Table 28: Subgraph expansion of M_4

The desired result follows. ■

We offer 2 different ways of obtaining a formula for the flow polynomial of the general cycle graph M_n.

Our first method is recursion: SRF applied to $M_n(a_1, a_2, \ldots, a_n)$ results in $M_{n-1}(a_1, a_2, \ldots, a_{n-1})$ and a $P_n(a_1, a_2, \ldots, a_{n-1})$, both of which can be assumed to have previously computed flow polynomials. In this manner, after applying SRF, we arrive at a formula for the flow polynomial of M_n which is in terms of M_{n-1} and X_i for $i \leq n-1$. Based on this argument, we state the following theorem:

Theorem 6.3 *Let C_n, the cycle of length n, be the underlying simple graph of the graph M_n whose edge multiplicities are $a_1, a_2, \ldots, a_n$. Then*

$$\begin{aligned} F(M_n, \omega) &= (-1)^{a_n} \left[\frac{\omega^{a_n} - 1}{1 - \omega} F(M_{n-1}, \omega) + F(P_n, \omega) \right] \\ &= (-1)^{a_n} \left[\frac{\omega^{a_n} - 1}{1 - \omega} F(M_{n-1}, \omega) + \prod_{i=1}^{n-1} F(X_{a_i}, \omega) \right]. \end{aligned}$$

The second way is to get more inspiration from Lemmas 6.1 and 6.2 by studying the columns for the powers of ω in detail. To make this task easier we make the following definition:

Definition 6.4 *Let the n-element set* $A = \{a_i\}_{i=1}^n$ *be given. Take the collection of all* $(j-1)$*-subsets of* A*, denoted by* Ω_{j-1}*. Consider any member of* Ω_{j-1}*, say* $\left\{a_{i_1}, a_{i_2}, \ldots, a_{i_{j-1}}\right\}$*. Let the sum of the elements of this subset be exponent of* ω*, i.e.,* $\omega^{a_{i_1}+a_{i_2}+\ldots+a_{i_{j-1}}}$*. We do this for all members of* Ω_{j-1} *and sum the resulting powers of* ω*. We call this sum* $\Psi(n, j-1, \omega)$*.*

Let the cycle of length n be the underlying simple graph of the graph $M_n = M_n(a_1, a_2, \ldots, a_n)$ whose edge multiplicities are $a_1, a_2, \ldots, a_n$. Then

$$\begin{aligned} F(M_n, \omega) &= \frac{(-1)^{\sum_{i=1}^n a_i}}{(1-\omega)^n} \left[\sum_{i=1}^{n-1} \left((-1)^{i+1} (\omega - \omega^{n+1-i}) \Psi(n, j-1, \omega) \right) \right. \\ &+ \left. (1-\omega)\omega^{\sum_{i=1}^n a_i} \right] \end{aligned} \tag{6.4}$$

Example 6.5 *Let us find the flow polynomial of* M_6*. Using Equation 6.4, first we determine all of the* $\binom{6}{0}$*=1 0-subsets,* $\binom{6}{1}$*=6 1-subsets,* $\binom{6}{2}$*=15 2-subsets,* $\binom{6}{3}$*=20 3-subsets,* $\binom{6}{4}$*=15 4-subsets. Next we find* $\Psi(6, j-1, \omega)$ *for* $j = 1, 2, 3, 4$*.*

$$\begin{aligned}
\Psi(6,0,\omega) &= \omega^0 = 1 \\
\Psi(6,1,\omega) &= \omega^a + \omega^b + \omega^c + \omega^d + \omega^e + \omega^f \\
\Psi(6,2,\omega) &= \omega^{a+b} + \omega^{b+c} + \omega^{c+d} + \omega^{d+e} + \omega^{e+f} + \omega^{f+a} + \omega^{a+c} + \omega^{b+d} \\
&+ \omega^{c+e} + \omega^{d+f} + \omega^{e+a} + \omega^{f+b} + \omega^{a+d} + \omega^{b+e} + \omega^{c+f} \\
\Psi(6,3,\omega) &= \omega^{a+b+c} + \omega^{b+c+d} + \omega^{c+d+e} + \omega^{d+e+f} + \omega^{e+f+a} \\
&+ \omega^{a+b+d} + \omega^{b+c+e} + \omega^{c+d+f} + \omega^{d+e+a} + \omega^{e+f+b} \\
&+ \omega^{a+b+e} + \omega^{b+c+f} + \omega^{c+d+a} + \omega^{d+e+b} + \omega^{e+f+c} \\
&+ \omega^{a+c+e} + \omega^{b+d+f} + \omega^{f+a+b} + \omega^{f+a+c} + \omega^{f+a+d} \\
\Psi(6,4,\omega) &= \omega^{a+b+c+d} + \omega^{b+c+d+e} + \omega^{c+d+e+f} + \omega^{d+e+f+a} \\
&+ \omega^{f+a+b+c} + \omega^{a+b+c+e} + \omega^{b+c+d+f} + \omega^{c+d+e+a} \\
&+ \omega^{e+f+a+c} + \omega^{f+a+b+d} + \omega^{a+b+d+e} + \omega^{b+c+e+f} \\
&+ \omega^{e+f+a+b} + \omega^{d+e+f+b} + \omega^{c+d+f+a}
\end{aligned}$$

We can now use the above Ψ *values with* $n = 6$ *in Equation 6.4 to obtain*

$$\begin{aligned} F(M_6, \omega) = \frac{(-1)^{a+b+c+d+e+f}}{(1-\omega)^6} &\left[(\omega - \omega^6)\Psi(6,0,\omega) \right. \\ - \quad & (\omega - \omega^5)\Psi(6,1,\omega) + (\omega - \omega^4)\Psi(6,2,\omega) - (\omega - \omega^3)\Psi(6,3,\omega) \\ + \quad & \left. (\omega - \omega^2)\Psi(6,4,\omega) + (1-\omega)\omega^{a+b+c+d+e+f} \right]. \end{aligned}$$

Finally, The proof of the following two results can be found in [7].

Theorem 6.6 *Let W_n, the wheel on $n+1$ vertices, be the underlying simple graph of the graph G, where the rim edges of G have multiplicity 1 and the spokes of G have edge multiplicities $\vec{a} = (a_1, a_2, \ldots, a_n)$. Pick any $\sigma \in S_n$ and apply σ to the spokes of G and call the new graph G_σ whose edge multiplicities now are $\sigma(\vec{a}) = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)})$. Then the flow polynomial of G is permutation invariant, i.e.,*

$$F(G, \lambda) = F(G_\sigma, \lambda).$$

Theorem 6.7 *Let C_n be the underlying simple graph of the graph G whose edge multiplicities are $\vec{a} = (a_1, a_2, \ldots, a_n)$. Pick any $\sigma \in S_n$ and apply σ to the edge bundles of G and call the new graph G_σ, whose edge multiplicities now are $\sigma(\vec{a}) = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)})$. Then the flow polynomial of G is permutation invariant, i.e.,*

$$F(G, \omega) = F(G_\sigma, \omega).$$

References

[1] Jaeger, F., "Even subgraph expansion for the flow polynomial of cubic plane maps", *J. Combinat. Theory, Ser. B* **52**, 1991, p. 259–273.

[2] Jaeger, F., "On nowhere-zero flows in multigraphs", *Proceedings, Fifth British Combinatorial Conferece, Aberdeen,* 1975, p. 373–378.

[3] Read, R. C., Tutte, W. T., "Chromatic polynomials", *in Selected Topics in Graph Theory 3, Academic Press*, 1988, p. 15–42.

[4] Read, R. C., Whitehead, E. G. Jr, "Chromatic polynomials of homeomorphism classes of graphs", Discrete Math., **204**(1999), p. 337–356.

[5] Seymour, P. D., "Nowhere-zero 6-flows", *J. Combin. Theory Ser. B.*, **30**, 1981, p. 130–135.

[6] Shahmohamad, H., "On minimum flow number of graphs", To appear in Bulletin of Institute of Combinatorics and Applications

[7] Shahmohamad, H., "On nowhere-zero flows, chromatic equivalence and flow equivalence of graphs", PhD Thesis, University of Pittsburgh, 2000.

[8] Tutte, W. T. , Graph Theory, *Addison-Wesley, Reading, Mass.*, 1984.

[9] Zhang, C. Q. , Integer Flows and Cycle Covers of Graphs, *Marcel Dekker, Inc.*, 1997.