ON PURELY PERIODIC NEAREST SQUARE CONTINUED FRACTIONS

KEITH MATTHEWS AND JOHN ROBERTSON

Abstract. We present a test for determining whether a real quadratic irrational has a purely periodic nearest square continued fraction expansion. This test is somewhat more explicit than the standard test and simplifies the programming of the algorithm.

1. Introduction

Simple tests have long been known for determining whether a real quadratic irrational $\xi = (P + \sqrt{D})/Q, D > 0$ and non–square, has a purely periodic regular continued fraction expansion, or nearest integer continued fraction expansion. Thus if $\overline{\xi} = (P - \sqrt{D})/Q$, then ξ has a purely periodic regular continued fraction expansion if and only if $\xi > 1$ and $-1 < \overline{\xi} < 0$ ([4, p. 22]). Also ξ has a purely periodic nearest integer continued fraction expansion if and only if $\xi > 2$ and $(1 - \sqrt{5})/2 < \overline{\xi} < (3 - \sqrt{5})/2$ ([3]).

No test for pure periodicity as simple as these is known for the *nearest square* continued fraction, defined in the next section. Instead, A. A. K. Ayyangar [2, p. 27] gave a definition of *reduced* quadratic irrational and showed that ξ has a purely periodic nearest square continued fraction expansion if and only if ξ is reduced. In this paper, we give a more explicit version of Ayyangar's definition which is useful in detecting the start of a period.

2. The Nearest square continued fraction algorithm

This continued fraction was introduced by A.A.K. Ayyangar in 1940 and 1941 (see [1], [2]). Let $\xi_0 = \frac{P+\sqrt{D}}{Q}$ be a surd in standard form, i.e.,

(i) P, Q and D are integers, $Q \neq 0$, D is not a perfect square,
(ii) $(P^2 - D)/Q$ is an integer,
(iii) $\gcd(P, Q, (D - P^2)/Q) = 1$.

Then with $c = \lfloor \xi_0 \rfloor$, the integer part of ξ_0, we can represent ξ_0 in two ways (the *positive* and *negative* representations of ξ_0):

$$\xi_0 = c + \frac{Q'}{P' + \sqrt{D}} = c + 1 - \frac{Q''}{P'' + \sqrt{D}},$$

where $\frac{P'+\sqrt{D}}{Q'} > 1$ and $\frac{P''+\sqrt{D}}{Q''} > 1$ are also in standard form. We choose the partial denominator a_0 and numerator ϵ_1 of the new continued fraction expansion as follows:

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(a) $a_0 = c$ if $|Q'| < |Q''|$, or $|Q'| = |Q''|$ and $Q < 0$, $\epsilon_1 = 1$,
(b) $a_0 = c+1$ if $|Q'| > |Q''|$, or $|Q'| = |Q''|$ and $Q > 0$, $\epsilon_1 = -1$.

The term *nearest square* arises on noting that $P'^2 = D - QQ'$ and $P''^2 = D + QQ''$ and restating (a) and (b) using the following equivalence:

$$|Q'| \geq |Q''| \iff |QQ'| \geq |QQ''| \iff |P'^2 - D| \geq |P''^2 - D|.$$

Then $\xi_0 = a_0 + \frac{\epsilon_1}{\xi_1}$, where $|\epsilon_1| = 1, a_0$ is an integer and $\xi_1 = \frac{P_1+\sqrt{D}}{Q_1} > 1$. Also $P_1 = P'$ or P'' and $Q_1 = Q'$ or Q'', according as $\epsilon_1 = 1$ or -1. We call ξ_1 the *successor* of ξ_0. We proceed similarly with ξ_1, and so on. Then the complete quotients ξ_n satisfy

$$\xi_n = a_n + \frac{\epsilon_{n+1}}{\xi_{n+1}} \text{ and } \xi_0 = a_0 + \frac{\epsilon_1}{|a_1} + \frac{\epsilon_2}{|a_2} + \cdots + \frac{\epsilon_n}{|\xi_n}, \tag{2.1}$$

with partial numerator $\epsilon_{i+1} = \pm 1$ and partial denominator $a_i \geq 1$ if $i \geq 1$. This expansion is called the *nearest square* continued fraction (NSCF) expansion.

Analogous relations to those for regular continued fractions also hold for P_n, Q_n and a_n:

$$P_{n+1} + P_n = a_n Q_n, \tag{2.2}$$

$$P_{n+1}^2 + \epsilon_{n+1} Q_n Q_{n+1} = D. \tag{2.3}$$

Ayyangar proved that the NSCF expansion is eventually periodic, i.e., the complete quotients ξ_n eventually satisfy $\xi_i = \xi_{i+k}$ for $i \geq i_0$ for some $k \geq 1$. Then $\epsilon_{i+1} = \epsilon_{i+k+1}$ and $a_i = a_{i+k}$ for all $i \geq i_0$ (see Theorem II, [2, p. 25]).

Our main result is the following:

Theorem. Let $\xi = (P + \sqrt{D})/Q$ be in standard form and let $R = (D - P^2)/Q$. Then ξ has a purely periodic nearest square continued fraction expansion if and only if

(i) $Q^2 + \frac{1}{4}R^2 \leq D, \quad \frac{1}{4}Q^2 + R^2 \leq D$,
(ii) ξ is the successor of $1/\xi$,
(iii) ξ is not of the form $\frac{p+q+\sqrt{p^2+q^2}}{2q}, p > 2q > 0$.

3. Reduced NSCF quadratic surds

Ayyangar [2, p. 27] gave the following definition of reduced quadratic surd. He first defined a *special* surd ξ_v by the inequalities

$$Q_{v+1}^2 + \tfrac{1}{4}Q_v^2 \leq D, \quad Q_v^2 + \tfrac{1}{4}Q_{v+1}^2 \leq D, \tag{3.1}$$

then defined a *semi-reduced* surd to be the successor of a special surd. Finally a *reduced* surd is defined to be the successor of a semi-reduced surd. Ayyangar proved ([2, p. 28]) that a semi–reduced surd is a special surd. Consequently a reduced surd is also semi–reduced. That a quadratic surd has a purely periodic NSCF expansion if and only if it is reduced, is proved in [2, pp. 101-102]. To use the Ayyangar characterization to decide if a surd ξ is reduced, one has to determine if there is a special surd whose second successor is ξ; doing this can be combersome.

One example (Theorem XII, [2, p. 102]) is the successor of $\sqrt{D}/Q > 1$, where Q divides D. Another example that figures prominently in [2] is $\frac{p+q+\sqrt{p^2+q^2}}{p}$, where $p > 2q > 0$.

Lemma 1. *If two different semi–reduced surds have the same successor, they have the form* $\frac{p\pm q+\sqrt{D}}{2q}$, *where* $p>2q>0$.

Proof. This is Theorem IX, [2, page 99]. □

Lemma 2. ξ *is semi–reduced if and only if* ξ *is reduced, or* $\xi=\frac{p+q+\sqrt{p^2+q^2}}{2q}, p>2q>0$.

Proof. (a) Suppose ξ is semi–reduced and let η be its successor. Then η is reduced and has a unique reduced predecessor χ, by Corollary 1, [2, p. 101]. By Lemma 1, either $\xi=\chi$, or ξ and χ are equal to $\frac{p\pm q+\sqrt{D}}{2q}$, where $p>2q>0$. However by Theorem X, [2, p. 100], $\frac{p+q+\sqrt{p^2+q^2}}{2q}$ has no semi–reduced predecessor and hence is not reduced, so $\chi=\frac{p-q+\sqrt{D}}{2q}$ and hence $\xi=\frac{p+q+\sqrt{p^2+q^2}}{2q}$.

(b) If ξ is reduced, it is the successor of a semi–reduced surd χ and as previously observed, this is special. Hence ξ is semi–reduced. If $\xi=\frac{p+q+\sqrt{p^2+q^2}}{2q}, p>2q>0$, in view of the equation

$$\frac{p-q+\sqrt{p^2+q^2}}{p}=1+\frac{p}{q+\sqrt{p^2+q^2}}=2-\frac{2q}{p+q+\sqrt{p^2+q^2}},$$

as $p>2q>0$, we see ξ is the successor of the special surd $\frac{p-q+\sqrt{p^2+q^2}}{p}$ and is hence semi–reduced. □

Corollary 1. ξ *is reduced if and only if* ξ *is semi–reduced and not of the form* $\xi=\frac{p+q+\sqrt{p^2+q^2}}{2q}, p>2q>0$.

Ayyangar did not explicitly mention Lemma 2 or Corollary 1 in his paper [2].

4. Some lemmas on successors

Lemma 3. ξ *and* $-\xi$ *have the same successor.*

Proof. This follows from the fact that the positive–negative represention

$$\frac{P+\sqrt{D}}{Q}=c+\frac{Q'}{P'+\sqrt{D}}=c+1-\frac{Q''}{P''+\sqrt{D}},$$

implies the positive–negative represention

$$\frac{P+\sqrt{D}}{-Q}=-c-1+\frac{Q''}{P''+\sqrt{D}}=-c-\frac{Q'}{P'+\sqrt{D}}.$$

Then the conditions defining the successor of ξ also define the same successor of $-\xi$.

□

Lemma 4. *If* ξ *is the successor of a quadratic surd, then* ξ *is the successor of* $1/\xi$.

Proof. Suppose $\xi=\frac{P+\sqrt{D}}{Q}$ is the successor of $\frac{P_0+\sqrt{D}}{Q_0}$. Then the successor equation

$$\frac{P_0+\sqrt{D}}{Q_0}=b+\epsilon\frac{Q}{P+\sqrt{D}} \tag{4.1}$$

gives $P_0 + P = bQ_0$ and $D - P^2 = \epsilon Q Q_0$. We also have the positive–negative representation

$$\frac{P_0 + \sqrt{D}}{Q_0} = a + \frac{Q'}{P' + \sqrt{D}} = a + 1 - \frac{Q''}{P'' + \sqrt{D}}.$$

Then

$$\begin{aligned}\epsilon/\xi = \frac{\epsilon Q}{P + \sqrt{D}} &= \frac{\epsilon Q(\sqrt{D} - P)}{D - P^2} = \frac{\epsilon Q(\sqrt{D} - P)}{\epsilon Q Q_0} \\ &= \frac{-P + \sqrt{D}}{Q_0} \qquad (4.2) \\ &= \frac{P_0 - bQ_0 + \sqrt{D}}{Q_0} \\ &= -b + \frac{P_0 + \sqrt{D}}{Q_0} \\ &= -b + a + \frac{Q'}{P' + \sqrt{D}} = -b + a + 1 - \frac{Q''}{P'' + \sqrt{D}}\end{aligned}$$

and this positive–negative representation implies that the successor of ϵ/ξ is ξ. If $\epsilon = -1$, Lemma 3 implies that the successor of $1/\xi$ is also ξ.

Lemma 5. *If* $\xi = \frac{P+\sqrt{D}}{Q}$, *let* $R = (D - P^2)/Q$. *Then* ξ *is semi–reduced if and only if*

$$\text{(i)} \quad Q^2 + \tfrac{1}{4}R^2 \le D, \quad \tfrac{1}{4}Q^2 + R^2 \le D,$$
$$\text{(ii)} \quad \xi \textit{ is the successor of } 1/\xi.$$

Proof. (a) Suppose $\xi = \frac{P+\sqrt{D}}{Q}$ is semi–reduced. Then ξ is the successor of a special surd $\xi_0 = \frac{P_0+\sqrt{D}}{Q_0}$. Then with ϵ as in (4.1), as before, we have $R = (D - P^2)/Q = \epsilon Q_0$ and inequalities $Q_0^2 + \frac{1}{4}Q^2 \le D$, $\frac{1}{4}Q^2 + Q_0^2 \le D$ become

$$R^2 + \tfrac{1}{4}Q^2 \le D, \quad \tfrac{1}{4}Q^2 + R^2 \le D.$$

Also by Lemma 4, ξ is the successor of $1/\xi$.
(b) Suppose (i) and (ii) hold. Then as $1/\xi = \frac{-P+\sqrt{D}}{R}$, (i) and (ii) imply $1/\xi$ is special and that ξ is semi–reduced. □

Our Theorem then follows from Corollary 1 and Lemma 5.
We also mention the following useful result.

Lemma 6. *If* ξ *is semi–reduced, then* ξ *or* $\xi - 1$ *is also reduced in the regular continued fraction sense.*

Proof. By Corollary 1, [2, p. 30], we have $\xi > \frac{1+\sqrt{5}}{2} > 1$ and by Theorem III, [2, p.27], $-1 < \overline{\xi} < 1$. So if $-1 < \overline{\xi} < 0$, ξ is RCF–reduced. If $0 < \overline{\xi}$, let $\xi = \frac{P+\sqrt{D}}{Q}$. Then $0 < \frac{P-\sqrt{D}}{Q}$ and as $0 < Q < \sqrt{D}$ by Theorem 1 (iv), [2, p. 22], we have

$$2 < \frac{2\sqrt{D}}{Q} < \frac{P + \sqrt{D}}{Q} = \xi.$$

Hence $1 < \xi - 1$ and as $-1 < \overline{\xi - 1} < 0$, it follows that $\xi - 1$ is RCF–reduced. □

We conclude with an example of Ayyangar in his Theorem XIII, [2, p. 103]. His proof of case (b), when $p > 2q$, involved a complicated discussion of inequalities.

Example. Let $\xi = \frac{P+\sqrt{D}}{Q} = \frac{p+q+\sqrt{p^2+q^2}}{p}, p > 0, q > 0$. Then

(a) ξ is not reduced if $p < 2q$,
(b) ξ is reduced if $p \geq 2q$,

Proof. Here $P = p + q, Q = p, R = (D - P^2)/Q = -2q$ and

$$Q^2 + \tfrac{1}{4}R^2 = p^2 + q^2 = D \text{ and } \tfrac{1}{4}Q^2 + R^2 = \tfrac{1}{4}p^2 + 4q^2.$$

Hence

$$\begin{aligned} \tfrac{1}{4}Q^2 + R^2 \leq D &\iff \tfrac{1}{4}p^2 + 4q^2 \leq p^2 + q^2 \\ &\iff 3q^2 \leq 3p^2/4 \\ &\iff 2q \leq p. \end{aligned}$$

Consequently if $p < 2q$, ξ is not reduced. However if $p \geq 2q$, then $\frac{1}{4}Q^2 + R^2 \leq D$. Also the positive–negative representation

$$\begin{aligned} 1/\xi &= \frac{-P + \sqrt{D}}{R} = \frac{-p - q + \sqrt{p^2 + q^2}}{-2q} \\ &= 0 + \frac{p}{p + q + \sqrt{p^2 + q^2}} = 1 - \frac{p}{p - q + \sqrt{p^2 + q^2}}, \end{aligned}$$

shows that ξ is the successor of $1/\xi$. Hence conditions (i) and (ii) of our Theorem are satisfied, so ξ is semi–reduced. Also condition (iii) is satisfied. For assume $\xi = \frac{P+Q+\sqrt{P^2+Q^2}}{2Q}, P > 2Q > 0$. Then

$$p + q = P + Q, p = 2Q, p^2 + q^2 = P^2 + Q^2.$$

Hence

$$\begin{aligned} P^2 + Q^2 &= 4Q^2 + q^2 \\ P^2 &= 3Q^2 + q^2 = 3Q^2 + (P - Q)^2 \\ &= 4Q^2 + P^2 - 2PQ \\ 2PQ &= 4Q^2 \\ P &= 2Q. \end{aligned}$$

This contradiction completes the demonstration that ξ is reduced if $p \geq 2q > 0$. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF QUEENSLAND, BRISBANE, AUSTRALIA, 4072 AND CENTRE FOR MATHEMATICS AND ITS APPLICATIONS, AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA, ACT 0200, AUSTRALIA

E-mail address: `keithmatt@gmail.com`

ACTUARIAL AND ECONOMIC SERVICES DIVISION, NATIONAL COUNCIL ON COMPENSATION INSURANCE, BOCA RATON, FL 33487
E-mail address: jpr2718@gmail.com