A nearest integer formula

If $y \in \mathbb{R}$, we define $[y]$, the nearest integer to y by $-1/2 \leq y - [y] < 1/2$.
In particular, if $y = z + 1/2, z \in \mathbb{Z}$, then $[y] = z + 1$.
It is well-known that $[y] = \lfloor y + 1/2 \rfloor$, where $\lfloor t \rfloor$ is the integer part of t.

Theorem. If $x \in \mathbb{R}, 2x \notin \mathbb{Z}$ and $m \in \mathbb{Z}, m \neq 0$, then

$$[x/m] = \begin{cases} \lfloor \frac{\lfloor x \rfloor + \lfloor \frac{m+F}{2} \rfloor}{m} \rfloor & \text{if } m > 0 \\ \lfloor \frac{1+\lfloor x \rfloor + \lfloor \frac{m+F+1}{2} \rfloor}{m} \rfloor & \text{if } m < 0, F+m+1 \notin 2\mathbb{Z} \\ \lfloor \frac{\lfloor x \rfloor + \lfloor \frac{m+F+1}{2} \rfloor}{m} \rfloor & \text{if } m < 0, F+m+1 \in 2\mathbb{Z} \end{cases}$$

where

$$F = \begin{cases} 1 & \text{if } x - \lfloor x \rfloor > 1/2 \\ 0 & \text{if } x - \lfloor x \rfloor < 1/2. \end{cases}$$

Remark. In view of the above, the formula for T_k on page 373 on page 373 of *Calculation of the Regulator of* $\mathbb{Q}\sqrt{D})$*, by Use of the Nearest Integer Continued Fraction Algorithm*, H.C. Williams and P.A. Buhr, Math. Comp. 33 (1979) 369-381, should be changed if $Q'_k < 0$ to

$$T_k = \begin{cases} 1 + d + \lfloor \frac{|Q'_k|+F+1}{2} \rfloor & \text{if } F + Q'_k + 1 \notin 2\mathbb{Z} \\ d + \lfloor \frac{|Q'_k|+F+1}{2} \rfloor & \text{if } F + Q'_k + 1 \in 2\mathbb{Z}. \end{cases}$$

Moreover John Robertson has subsequently pointed out that the definition of R'_{k+1} at the bottom of page 373 should be amended when $Q'_{k+1} < 0$ and $Q'_{k+1} | P_{k+1} + T_{k+1}$: we have to define $R'_{k+1} = -Q'_{k+1}$.

Lemma. If $x \in \mathbb{R}$ and $m \in \mathbb{Z}$, then

$$\lfloor x/m \rfloor = \begin{cases} \lfloor \frac{\lfloor x \rfloor}{m} \rfloor & \text{if } m > 0 \\ \lfloor \frac{1+\lfloor x \rfloor}{m} \rfloor & \text{if } m < 0 \text{ and } x \notin \mathbb{Z}. \end{cases}$$

Proof of Theorem. First note that $\lfloor 2x \rfloor = 2\lfloor x \rfloor + F$.
Case 1. Assume $m > 0$. Then

$$\begin{aligned} \left\lfloor \frac{x}{m} + \frac{1}{2} \right\rfloor &= \left\lfloor \frac{2x+m}{2m} \right\rfloor \\ &= \left\lfloor \frac{\lfloor 2x+m \rfloor}{2m} \right\rfloor \text{ (by the Lemma)} \end{aligned}$$

$$
\begin{aligned}
&= \left\lfloor \frac{\lfloor 2x \rfloor + m}{2m} \right\rfloor \\
&= \left\lfloor \frac{2\lfloor x \rfloor + F + m}{2m} \right\rfloor \\
&= \left\lfloor \frac{\lfloor x \rfloor + \frac{F+m}{2}}{m} \right\rfloor \\
&= \left\lfloor \frac{\lfloor x \rfloor + \lfloor \frac{F+m}{2} \rfloor}{m} \right\rfloor \text{ (by the Lemma).}
\end{aligned}
$$

Case 2. Assume $m < 0$ and $2x \notin \mathbb{Z}$. Then

$$
\begin{aligned}
\left\lfloor \frac{x}{m} + \frac{1}{2} \right\rfloor &= \left\lfloor \frac{2x + m}{2m} \right\rfloor \\
&= \left\lfloor \frac{1 + \lfloor 2x + m \rfloor}{2m} \right\rfloor \\
&= \left\lfloor \frac{\lfloor 2x \rfloor + m + 1}{2m} \right\rfloor \\
&= \left\lfloor \frac{2\lfloor x \rfloor + F + m + 1}{2m} \right\rfloor \\
&= \left\lfloor \frac{\lfloor x \rfloor + (F + m + 1)/2}{m} \right\rfloor \\
&= \begin{cases} \left\lfloor \frac{1+\lfloor x \rfloor + \lfloor (F+m+1)/2 \rfloor}{m} \right\rfloor & \text{if } F + m + 1 \notin 2\mathbb{Z} \\ \left\lfloor \frac{\lfloor x \rfloor + \lfloor (F+m+1)/2 \rfloor}{m} \right\rfloor & \text{if } F + m + 1 \in 2\mathbb{Z}. \end{cases}
\end{aligned}
$$