On the converse of a theorem of Nagell and Tchebicheff

Keith Matthews[a], John Robertson[1]

[a]*Department of Mathematics*
University of Queensland
Brisbane, Australia, 4072
and
Centre for Mathematics and its Applications
Australian National University
Canberra, ACT, Australia, 0200
[b]*Actuarial and Economic Services Division*
National Council on Compensation Insurance
Boca Raton, FL 33487

Abstract

A well–known upper estimate for the size of the fundamental solutions of the diophantine equation $u^2 - Dv^2 = N$ was proved by Tchebicheff and rediscovered a century later by Nagell. P. G. Tsangaris has proved a converse in his PhD thesis and stated the result in [11], that any solution which satisfies this inequality is indeed a fundamental solution. Surprisingly this is the only place we have seen the converse stated. We give a proof and also show that the ambiguous classes correspond to equality in these estimates. Finally we mention some related results of Frattini and Tsangaris on the determination of the non–negative solutions and give an efficient algorithm based on continued fractions, for finding the least non–negative primitive solutions in each class.

Keywords: Quadratic diophantine equation, fundamental solution, ambiguous class, continued fraction
2000 MSC: 11D09

Email addresses: keithmatt@gmail.com (Keith Matthews), jpr2718@gmail.com (John Robertson)

1. Introduction

We consider the diophantine equation

$$u^2 - Dv^2 = N, \tag{1}$$

where $D > 0$ is not a perfect square and N is nonzero.

Following Nagell [8], two integer solutions $(u, v), (u', v')$ of equation (1) are called *equivalent* if

$$u' + v'\sqrt{D} = (u + v\sqrt{D})(x + y\sqrt{D}),$$

where (x, y) satisfies Pell's equation $x^2 - Dy^2 = 1$. The equivalence classes come in pairs, with (u, v) and $(-u, v)$ in general defining different classes. If (u, v) and $(-u, v)$ define the same class, this class is called *ambiguous*. Among all solutions $u + v\sqrt{D}$ in a class K, we choose a solution $u^* + v^*\sqrt{D}$, where v^* is the least non–negative value of v when $u + v\sqrt{D}$ belongs to K. Equivalently $|u^*|$ is the least value of $|u|$ when $u + v\sqrt{D}$ belongs to K. In the case of an ambiguous class, we choose $u^* \geq 0$. There are finitely many equivalence classes, each indexed by a *fundamental* solution (u^*, v^*). In his book [8] and paper [7], Nagell gave the following necessary conditions for $u + v\sqrt{D}$ to be a fundamental solution.

Proposition 1. (Nagell–Tchebicheff) *Suppose $x_1 + y_1\sqrt{D}$ is the least positive solution of Pell's equation and let $u + v\sqrt{D}$ be a fundamental solution of the equation $u^2 - Dv^2 = N$. Then u and v satisfy the following inequalities:*

(a) *If $N > 0$, then*

$$0 \leq v \leq y_1\sqrt{\tfrac{N}{2(x_1+1)}}, \tag{2}$$

$$\sqrt{N} \leq |u| \leq \sqrt{\tfrac{1}{2}(x_1 + 1)N}. \tag{3}$$

(b) *If $N < 0$, then*

$$\sqrt{\tfrac{|N|}{D}} \leq v \leq y_1\sqrt{\tfrac{|N|}{2(x_1-1)}}, \tag{4}$$

$$0 \leq |u| \leq \sqrt{\tfrac{1}{2}(x_1 - 1)|N|}. \tag{5}$$

In each set of inequalities, we have equality on the lower (upper) bounds for u if and only if we have equality on the lower (upper) bounds for v.

Nagell was apparently unaware that he had been anticipated by Tchebicheff [10] in 1851. Nagell's necessary conditions are also sufficient. This was not stated explicitly by Nagell and Tchebicheff, perhaps because the essential step in proving sufficiency is simply a matter of reversing Nagell's proof. The sufficiency part is stated by P. G. Tsangaris in Theorem 1.1, [11]. G. Frattini [1], like Tsangaris, was primarily interested in finding all positive integer solutions of (1). Frattini was aware of Tchebicheff's work and wrote extensively about (9). For the record, we give proofs of the converse of Nagell's Theorem and also describe the ambiguous classes explicitly. We state Frattini's main results about non–negative integer solutions of (1) in section 4. Finally we point out that an efficient way of finding the smallest non–negative solutions of (1) is given by using continued fractions.

Theorem 1. (Converse of Nagell–Tchebicheff) *Suppose $x_1 + y_1\sqrt{D}$ is the least positive solution of Pell's equation and let $u + v\sqrt{D}$ be a fundamental solution of the equation $u^2 - Dv^2 = N$. Then u and v satisfy the following inequalities:*

(a) *If $N > 0$ and $0 \le v \le y_1\sqrt{\frac{N}{2(x_1+1)}}$, with $u > 0$ in the case of equality, then $u + v\sqrt{D}$ is a fundamental solution.*

(b) *If $N < 0$ and $0 < v \le y_1\sqrt{\frac{|N|}{2(x_1-1)}}$, with $u > 0$ in the case of equality, then $u + v\sqrt{D}$ is a fundamental solution.*

Nagell did not state the following result which characterises the ambiguous classes.

Theorem 2. *Suppose $u + v\sqrt{D}$ is a fundamental solution of $u^2 - Dv^2 = N$ with $u \ge 0$ and $v \ge 0$. Then $u+v\sqrt{D}$ defines an ambiguous class if and only if one of* (a) *and* (b) *holds:*

(a) $N > 0, u > 0$ *and* $v = 0$ *or* $v = y_1\sqrt{\frac{N}{2(x_1+1)}}$;

(b) $N < 0, v > 0$ *and* $u = 0$ *or* $v = y_1\sqrt{\frac{|N|}{2(x_1-1)}}$.

Consequently there are at most two ambiguous classes.

2. Proof of Theorem 1

PROOF. The equivalence class K defined by $u+v\sqrt{D}$ consists of the numbers $u'+v'\sqrt{D} = \pm(u+v\sqrt{D})\epsilon^n, n \in \mathbb{Z}$, where $\epsilon = x_1+y_1\sqrt{D}$ is the least solution of Pell's equation. The fact that $u+v\sqrt{D}$ is a fundamental solution for class K will follow from the inequality $|v'| \geq v$. This is immediate in the following case:

$$(u + v\sqrt{D})\epsilon^n = r_n + s_n\sqrt{D}, \tag{6}$$

for $n \geq 0$, where $s_{n+1} > s_n \geq v$ and $r_{n+1} > r_n \geq u$. In particular, $v' = r_n \geq v$.

We now distinguish two cases: (a) $N > 0$, (b) $N < 0$.

Case (a) $N > 0$. We assume $u \geq 0$ and

$$0 \leq v \leq y_1\sqrt{\tfrac{N}{2(x_1+1)}}. \tag{7}$$

Then

$$0 < u \leq \sqrt{\tfrac{1}{2}(x_1 + 1)N}. \tag{8}$$

Also $u > 0$, as $u = 0$ implies $-Dv^2 = N$. We now prove

$$(u + v\sqrt{D})\epsilon^{-m} = u_m - v_m\sqrt{D}, \tag{9}$$

for $m \geq 1$, where $v_{m+1} > v_m \geq v$ and $u_{m+1} > u_m \geq u$. In particular, $v' = v_m \geq v$.

We first deal with the case $m = 1$.

$$\begin{aligned}(u + v\sqrt{D})\epsilon^{-1} &= (u + v\sqrt{D})(x_1 - y_1\sqrt{D}) \\ &= ux_1 - vy_1D - (uy_1 - vx_1)\sqrt{D} \\ &= u_1 - v_1\sqrt{D}.\end{aligned}$$

Reversing Nagell's argument, from (8) we deduce

$$\begin{aligned}u^2 \leq \frac{1}{2}(x_1 + 1)N &\implies \frac{x_1 - 1}{x_1 + 1} \geq 1 - \frac{N}{u^2} \\ &\implies u^2(x_1 - 1)^2 \geq (u^2 - N)(x_1^2 - 1) = D^2v^2y_1^2 \\ &\implies u(x_1 - 1) \geq Dvy_1 \\ &\implies u_1 = ux_1 - Dvy_1 \geq u.\end{aligned}$$

Also (7) implies

$$\begin{aligned} v^2 2(x_1+1) \le Ny_1^2 &\implies v^2(2x_1+1) + v^2(x_1^2 - Dy_1^2) \le Ny_1^2 \\ &\implies v^2(x_1^2 + 2x_1 + 1 \le (Dv^2+N)y_1^2 = u^2y_1^2 \\ &\implies v(x_1+1) \le uy_1 \\ &\implies v \le uy_1 - vx_1 = v_1. \end{aligned}$$

Then by induction on $m \ge 1$, using the recurrence relations

$$\begin{aligned} u_{m+1} &= u_m x_1 + Dv_m y_1, \\ v_{m+1} &= v_m x_1 + u_m y_1, \end{aligned}$$

we have $v_{m+1} > v_m \ge v$ and $u_{m+1} > u_m \ge u$.

For future reference, taking conjugates, equation (9) gives

$$(u - v\sqrt{D})\epsilon^m = u_m + v_m\sqrt{D}, \tag{10}$$

where $u_{m+1} > u_m \ge u$ and $v_{m+1} > v_n \ge v$ for $m \ge 1$.

Case (b) $N < 0$. We assume $u \ge 0$ and

$$0 \le v \le y_1\sqrt{\tfrac{N}{2(x_1-1)}}. \tag{11}$$

Then

$$0 \le u \le \sqrt{\tfrac{1}{2}(x_1-1)N}. \tag{12}$$

For $m \ge 1$, we now prove

$$(u + v\sqrt{D})\epsilon^{-m} = -U_m + V_m\sqrt{D}, \tag{13}$$

where $U_{m+1} > U_m \ge u$ and $V_{m+1} > V_m \ge v$. In particular, $v' = V_m \ge v$.

We first deal with the case $m = 1$. We have

$$\begin{aligned} (u + v\sqrt{D})\epsilon^{-1} &= (u + v\sqrt{D})(x_1 - y_1\sqrt{D}) \\ &= -(vy_1D - ux_1) + (vx_1 - uy_1)\sqrt{D} \\ &= -U_1 + V_1\sqrt{D}. \end{aligned}$$

Reversing Nagell's argument, from (12) we deduce

$$\begin{aligned} u^2 \le \frac{1}{2}(x_1-1)|N| &\implies \frac{x_1+1}{x_1-1} \le 1 - \frac{N}{u^2} \\ &\implies u^2(x_1+1)^2 \le (u^2+N)(x_1^2-1) = D^2v^2y_1^2 \\ &\implies u(x_1+1) \le Dvy_1 \\ &\implies u \le Dvy_1 - ux_1 = U_1. \end{aligned}$$

Also (11) implies

$$\begin{aligned}
v^2 2(x_1-1) \le |N|y_1^2 &\implies v^2(2x_1-1) - v^2(x_1^2 - Dy_1^2) \le \; -Ny_1^2 \\
&\implies \; -v^2(x_1^2 - 2x_1 + 1 \le (-Dv^2 - N)y_1^2 = -u^2y_1^2 \\
&\implies v(x_1-1) \le uy_1 \\
&\implies V_1 = vx_1 - uy_1 \ge v.
\end{aligned}$$

Then by induction on $m \ge 1$, using the recurrence relations

$$\begin{aligned}
U_{m+1} &= U_m x_1 + DV_m y_1, \\
V_{m+1} &= V_m x_1 + U_m y_1,
\end{aligned}$$

we have $U_m \ge u$ and $V_m \ge v$.

For future reference, taking conjugates, equation (14) gives

$$(-u + v\sqrt{D})\epsilon^m = U_m + V_m\sqrt{D}, \tag{14}$$

where $U_{m+1} > U_m \ge u$ and $V_{m+1} > V_n \ge v$ for $m \ge 1$.

3. Proof of Theorem 2

Proof. Suppose $u+v\sqrt{D}$ is a fundamental solution of $u^2 - Dv^2 = N$ where $u \ge 0$.

(a) Suppose $N > 0$. Then $u > 0$. If $v = 0$, then $u+v\sqrt{D} = -(-u+v\sqrt{D})$ and $u + v\sqrt{D}$ is an ambiguous solution. If $v = y_1\sqrt{\frac{N}{2(x_1+1)}}$, we have

$$\begin{aligned}
Ny_1^2 &= v^2(2x_1+2) \\
-v^2 + Ny_1^2 &= v^2(2x_1+1) \\
v^2(Dy_1^2 - x_1^2) + Ny_1^2 &= v^2(2x_1+1) \\
(Dv^2 + N)y_1^2 &= v^2(x_1^2 + 2x_1 + 1) \\
u^2y_1^2 &= v^2(x_1+1)^2 \\
uy_1 &= v(x_1+1).
\end{aligned} \tag{15}$$

Also $x_1^2 - 1 = Dy_1^2$ and hence from (15),

$$u(x_1 - 1) = Dy_1 v. \tag{16}$$

Then (15) and (16) combine to give

$$(u+v\sqrt{D})(-x_1+y_1\sqrt{D}) = -u+v\sqrt{D}$$

and $u+v\sqrt{D}$ defines an ambiguous class.

(b) Suppose $N<0$ and $v>0$. If $u=0$, then $u+v\sqrt{D}=(-u+v\sqrt{D})$ and $u+v\sqrt{D}$ is an ambiguous solution. If $v=y_1\sqrt{\frac{N}{2(x_1-1)}}$, then as in (a), we deduce $uy_1=v(x_1-1), u(x_1+1)=Dy_1v$. Hence

$$(u+v\sqrt{D})(x_1-y_1\sqrt{D}) = -u+v\sqrt{D}$$

and $u+v\sqrt{D}$ defines an ambiguous class.

Conversely, suppose $u+v\sqrt{D}$ defines an ambiguous class. Then

$$(u+v\sqrt{D})(-X_1+Y_1\sqrt{D}) = -u+v\sqrt{D}, \tag{17}$$

where $X_1^2-DY_1^2=1$. Then (17) gives

$$-uX_1+vDY_1 = -u, \tag{18}$$
$$-vX_1+uY_1 = v \tag{19}$$

and hence $X_1=(u^2+Dv^2)/N$ and $Y_1=2uv/N$.

(a) Assume $N>0$. Then $u>0$ and inequalities (3) and (2) imply

$$uv \le \sqrt{\tfrac{(x_1+1)N}{2}}\cdot y_1\sqrt{\tfrac{N}{2(x_1+1)}} = Ny_1/2,$$

so $Y_1=2uv/N\le y_1$. Hence either $Y_1=0$ and so $v=0$, or $0<Y_1\le y_1$ and hence $Y_1=y_1$ and $X_1=x_1$. Then (19) gives $uy_1=v(x_1+1)$, which on squaring and then reversing the chain of equalities from (15), leads back to $v=y_1\sqrt{\frac{N}{2(x_1+1)}}$.

(b) Assume $N<0$. Then $v>0$ and inequalities (4) and (5) imply

$$uv \le \sqrt{\tfrac{(x_1-1)|N|}{2}}\cdot y_1\sqrt{\tfrac{|N|}{2(x_1-1)}} = |N|y_1/2,$$

so $|Y_1|=2uv/|N|\le y_1$. Hence either $Y_1=0$ and so $u=0$, or $0<|Y_1|\le y_1$, in which case $|Y_1|=y_1, Y_1=-y_1$ and $X_1=-x_1$. Then (19) gives $uy_1=v(x_1-1)$, which on squaring and then reversing a similar chain of equalities leading back from (15), gives $v=y_1\sqrt{\frac{|N|}{2(x_1-1)}}$.

Example 1. (Lagrange 1769, [9, 471–485]). The equation $u^2 - 46v^2 = 210$. Here $x_1 = 24335, y_1 = 3588$, so the fundamental solutions $u+v\sqrt{46}$ satisfy

$$0 \leq v \leq y_1\sqrt{\tfrac{N}{2(x_1+1)}} = 3588\sqrt{\tfrac{210}{2\cdot 24336}} = 235.67\cdots$$

We find solutions for $v = 1, 11, 43, 79$:

$$\pm 16 + \sqrt{46}, \quad \pm 76 + 11\sqrt{46}, \quad \pm 292 + 43\sqrt{46}, \quad \pm 536 + 79\sqrt{46}.$$

Example 2. (Frattini 1891, [1, p. 179]). The equation $u^2 - 13v^2 = -12$. Here $x_1 = 649, y_1 = 180$, so the fundamental solutions $u + v\sqrt{13}$ satisfy

$$0 \leq v \leq y_1\sqrt{\tfrac{N}{2(x_1-1)}} = 180\sqrt{\tfrac{12}{1296}} = 17.32\cdots$$

We find solutions for $v = 1, 4, 7$:

$$\pm 1 + \sqrt{13}, \quad \pm 14 + 4\sqrt{13}, \quad \pm 25 + 7\sqrt{13}.$$

Example 3. The equation $u^2 - 96v^2 = 4$. We have $x_1 = 49, y_1 = 5$ and the Nagell upper bound is $5\sqrt{\frac{4}{2\cdot 50}} = 1$. The fundamental solutions $u + v\sqrt{96}$ satisfy $0 \leq v \leq 1$ and we find ambiguous fundamental solutions $2 + 0\sqrt{96}$ and $10 + \sqrt{96}$.

Example 4. The equation $u^2 - 96v^2 = -96$. We have $x_1 = 49, y_1 = 5$ and the Nagell upper bound is $5\sqrt{\frac{96}{2\cdot 50}} = 5$. The fundamental solutions $u + v\sqrt{96}$ satisfy $0 \leq v \leq 5$ and we find ambiguous fundamental solutions $0 + \sqrt{96}$ and $48 + 5\sqrt{96}$.

4. The non–negative solutions

Frattini studied the non–negative integer solutions of $x^2 - Dy^2 = N$ in a series of papers which are listed in the Jahrbuch Database [4]. Using Frattini's notation, let $\alpha + \beta\sqrt{D}$ be the least positive solution of the Pell equation $\alpha^2 - D\beta^2 = 1$. In [1], Frattini gave a descent proof of the following results.

Theorem 3. (a) *The non–negative solutions of $x^2 - Dy^2 = N, N > 0$ are given by*

$$x + y\sqrt{D} = (k + h\sqrt{D})(\alpha + \beta\sqrt{D})^m, m \geq 0, \tag{20}$$

where (k, h) runs through the non–negative solutions of $k^2 - Dh^2 = N$ with $h < \beta\sqrt{N}$.

(b) *The non–negative solutions of* $x^2 - Dy^2 = -N, N > 0$ *are given by*

$$x + y\sqrt{D} = \begin{cases} (k + h\sqrt{D})(\alpha + \beta\sqrt{D})^m, & m \geq 0, \\ (-k + h\sqrt{D})(\alpha + \beta\sqrt{D})^m, & m \geq 1, \end{cases} \tag{21}$$

where (k, h) *runs through the non–negative solutions of* $k^2 - Dh^2 = -N$ *with* $h \leq \sqrt{\frac{N(\alpha+1)}{2D}} = \beta\sqrt{\frac{N}{2(\alpha-1)}}$.

Plainly, criterion (b) is much faster to implement computationally than (a). Also (b) is also stated as Theorem 2.3 in [11].

In [3] and the footnote on [2, p. 91], Frattini acknowledges Tchebicheff's upper estimates and remarks that the non–negative solutions of $x^2 - Dy^2 = N$, where $N > 0$, are also given by

$$x + y\sqrt{D} = \begin{cases} (k + h\sqrt{D})(\alpha + \beta\sqrt{D})^m, & m \geq 0, \\ (k - h\sqrt{D})(\alpha + \beta\sqrt{D})^m, & m \geq 1, \end{cases} \tag{22}$$

where (k, h) runs through the non–negative solutions of $k^2 - Dh^2 = N$ with $0 \leq h \leq \sqrt{\frac{N(\alpha-1)}{2D}}$. This is also stated as Theorem 2.4 in [11].

Both (21) and (22) were also stated as Proposition 1.2 in [11].

Frattini does not seem to explicitly state that the non–negative solutions of $x^2 - Dy^2 = -N, N > 0$ are also given by

$$x + y\sqrt{D} = (k + h\sqrt{D})(\alpha + \beta\sqrt{D})^m, m \geq 0, \tag{23}$$

where (k, h) runs through the non–negative solutions of $k^2 - Dh^2 = -N$ with $h < \beta\sqrt{N}$.

Example 5. For formula (21), Frattini gives the example $x^2 - 13y^2 = -12$ in [1, p. 179], where $\epsilon = 649 + 180\sqrt{13}$. Here $\sqrt{\frac{N(\alpha+1)}{2D}} = \sqrt{\frac{12\cdot 650}{2\cdot 13}} = 17.32\cdots$ and the positive solutions are given in terms of the fundamental solutions by

$$\begin{array}{llll} (1 + \sqrt{13})\epsilon^m, & (14 + 4\sqrt{13})\epsilon^m, & (25 + 7\sqrt{13})\epsilon^m, & m \geq 0, \\ (-1 + \sqrt{13})\epsilon^m, & (-14 + 4\sqrt{13})\epsilon^m, & (-25 + 7\sqrt{13})\epsilon^m, & m \geq 1, \end{array}$$

Alternatively, using formula (23), we have $\beta\sqrt{N} = 180\sqrt{12} = 623.53\cdots$ Then the positive solutions (k, h) where $0 \leq h \leq 623$ are given by:

$$(1, 1), \quad (14, 4), \quad (25, 7), \quad (155, 43), \quad (274, 76), \quad (1691, 469).$$

We get the last three solutions from the first three:

$$(-1+\sqrt{13})\epsilon = 1691+469\sqrt{13},$$
$$(-14+4\sqrt{13})\epsilon = 274+76\sqrt{13},$$
$$(-25+7\sqrt{13})\epsilon = 155+43\sqrt{13}.$$

Then the positive solutions of $x^2-13y^2=-12$ are given by

$$\begin{array}{llll}(1+\sqrt{13})\epsilon^m, & (14+4\sqrt{13})\epsilon^m, & (25+7\sqrt{13})\epsilon^m, & \\ (155+43\sqrt{13})\epsilon^m, & (274+76\sqrt{13})\epsilon^m, & (1691+469\sqrt{13})\epsilon^m, & m\geq 0.\end{array}$$

Example 6. For formula (22), Frattini gives the example $x^2-12y^2=52$ in [1, p. 179], where $\epsilon=7+2\sqrt{12}$. Here $\sqrt{\frac{N(\alpha-1)}{2D}}=\sqrt{\frac{52\cdot 6}{2\cdot 12}}=\sqrt{13}$, whose integer part is 3 and the positive solutions are given in terms of the fundamental solutions by

$$\begin{array}{lll}(8+\sqrt{13})\epsilon^m, & (10+2\sqrt{12})\epsilon^m, & m\geq 0,\\ (8-\sqrt{13})\epsilon^m, & (10-2\sqrt{12})\epsilon^m, & m\geq 1.\end{array}$$

Alternatively, the positive solutions $(k,h), h<\beta\sqrt{N}=2\sqrt{52}=14.42\cdots$ are given by formula (20):

$$(8,1),\quad (10,2),\quad (22,6),\quad (32,9).$$

We get the last two solutions from the first two:

$$(8-\sqrt{12})(7+2\sqrt{12}) = 32+9\sqrt{12},$$
$$(10-2\sqrt{12})(7+2\sqrt{12}) = 22+6\sqrt{12}.$$

Then the positive solutions of $x^2-12y^2=52$ are given by

$$(8+\sqrt{12})\epsilon^m,\quad (10+2\sqrt{12})\epsilon^m,\quad (22+6\sqrt{12})\epsilon^m,\quad (32+9\sqrt{12})\epsilon^m, m\geq 0.$$

5. Using continued fractions

A much more efficient method for finding the fundamental solutions uses continued fractions. This goes back to a neglected algorithm of Lagrange and was rediscovered by Matthews and Mollin . See reference [5]. We point out that the least non–negative solutions in each equivalence class can similarly be found. Clearly it suffices to consider *primitive* solutions of $x^2-dy^2=M$, where $M = N/f^2$, i.e., suppose (x, y) is a solution with $\gcd(x, y) = 1$ and $x > 0, y > 0$. Let $x \equiv yP \pmod{|M|}$. Then

$$P^2 \equiv d \pmod{|M|}. \tag{24}$$

Write $x = Py + |M|X$. Then substituting for x in $x^2 - dy^2 = M$ gives

$$|M|X^2 + 2PXy + (P^2 - d)y^2/|M| = M/|M|.$$

It can be proved that X/y is a convergent A_{n-1}/B_{n-1} of $\omega = (-P+\sqrt{d})/|M|$. See Theorem 1, [5, p. 325]. Then $x = |M|A_{n-1} + PB_{n-1} = G_{n-1}$, where (see Theorem 5.3.4 [6, p. 246])

$$M = x^2 - dy^2 = G_{n-1}^2 - dB_{n-1}^2 = (-1)^n|M|Q_n.$$

Hence $Q_n = (-1)^n M/|M|$.

The algorithm: For each solution $P, -|M|/2 < P \leq |M|/2$ of congruence (24), let $(P_n+\sqrt{d})/Q_n$ denote the nth complete quotient for ω. We search the continued fraction expansion of ω for the least n such that $Q_n = (-1)^n M/|M|$ and $G_{n-1} \geq 0$. Then (G_{n-1}, B_{n-1}) will be the least positive primitive solution of $x^2 - dy^2 = M$ for the class determined by P.

Example 7. Consider the equation $x^2 - 13y^2 = -12$ of Example 5. Here we have cases (a) $f = 1, M = -12$ and (b) $f = 2, M = -3$. We have to find the least positive primitive solutions of (a) $x^2 - 13y^2 = -12$ and (b) $x^2 - 13y^2 = -3$. For (a) the solutions P of $P^2 \equiv 13 \pmod{12}$ are $1, -1, 5, -5$.

P	$(-P + \sqrt{d})/\lvert M\rvert$	n	A_{n-1}/B_{n-1}	(G_{n-1}, B_{n-1})
5	$(-5 + \sqrt{13})/12$	7	$-5/43$	$(155, 43)$
-5	$(5 + \sqrt{13})/12$	5	$5/7$	$(25, 7)$
1	$(-1 + \sqrt{13})/12$	1	$0/1$	$(1, 1)$
-1	$(1 + \sqrt{13})/12$	11	$180/469$	$(1691, 469)$

giving least positive solutions $(155, 43), (25, 7), (1, 1), (1691, 469)$.

For (b) the solutions P of $P^2 \equiv 13 \pmod{3}$ are $1, -1$.

P	$(-P + \sqrt{d})/\lvert M\rvert$	n	A_{n-1}/B_{n-1}	(G_{n-1}, B_{n-1})
1	$(-1 + \sqrt{13})/3$	7	$33/38$	$(137, 38)$
-1	$(1 + \sqrt{13})/3$	3	$3/2$	$(7, 2)$

giving least positive solutions $(274, 76)$ and $(14, 4)$.

References

[1] G. Frattini, Dell'analisi indeterminata di secondo grado, Periodico di Mat. VI (1891) 169–180.

[2] G. Frattini, A complemento di alcuni teore mi del sig. Tchebicheff, Rom. Acc. L. Rend. 5 (1892) I No. 2, 85-91.

[3] G. Frattini, Dell'analisi indeterminata di secondo grado, Periodico di Mat. VII (1892) 172–177.

[4] The Jahrbuch Database, http://www.emis.de/MATH/JFM/JFM.html.

[5] K. R. Matthews, The Diophantine Equation $x^2 - Dy^2 = N, D > 0$, Expos. Math. 18 (2000), 323–331.

[6] R. A. Mollin, Fundamental Number Theory with Applications, CRC Press, NY 1998.

[7] T. Nagell, Bemerkung über die Diophantische Gleichung $u^2 - Dv^2 = C$, Arch. Math. 3 (1952), 8–9.

[8] T. Nagell, Introduction to number theory, Chelsea, 1981.

[9] J.-A. Serret (Ed), Oeuvres de Lagrange, I–XIV, Gauthiers–Villars, Paris 1877.

[10] M. P. Tchebichef, Sur les formes quadratiques, J. de math. pur. et appl. 16 (1851), 257–282.

[11] P. G. Tsangaris, Fermat–Pell equation and the numbers of the form $w^2 + (w + 1)^2$, Publ. Math. Debrecen 47 (1995), 127–138.