AN APPLICATION OF PERRON'S THEOREM 2.13

JOHN P. ROBERTSON

This note proves the theorem below, which was recently conjectured by Keith Matthews.

For the regular continued fraction (RCF) of a number ω, we write the partial fractions as a_i for $i \geq 0$, and we write the RCF of ω as $\omega = <a_0, a_1, \ldots>$. For $i \geq 1$, $a_i > 0$. We write the convergents to the RCF as A_i/B_i for $i \geq 0$. Following standard conventions, we set $A_{-1} = 1$, $A_0 = a_0$, $B_{-1} = 0$, and $B_0 = 1$ [1, §5, (3) and (3a), p. 10].

Theorem 1. *Let P, Q, R, S be integers so that $PS - QR = \pm 1$, $Q < 0$, and $S > -Q$. Also let $\varphi = (1 + \sqrt{5})/2$ and*

$$\omega = \frac{R\varphi + P}{S\varphi + Q}.$$

Then

(i) $\omega = <a_0, a_1, \ldots, a_r, \overline{1}>$, *where $r > 0$ and $a_r > 1$.*
(ii) $R = A_r - A_{r-1}$, $P = 2A_{r-1} - A_r$, $S = B_r - B_{r-1}$, *and* $Q = 2B_{r-1} - B_r$.

Proof. Perron's Theorem 2.13 [1, §13, Thm 2.13, p. 40] is (translated from the original German):

Theorem 2.13. *If*

$$\xi_0 = \frac{P\omega + R}{Q\omega + S}$$

where $\omega > 1$, and P, Q, R, S are integers satisfying the conditions

$$PS - QR = \pm 1, \quad Q > S > 0,$$

then R/S, P/Q are two successive convergents to ξ_0, and ω is the associated complete quotient.

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Henceforth we use the notation of Theorem 1, as opposed to that of the Perron's Theorem 2.13, unless otherwise specified.

Using $\varphi^2 = 1 + \varphi$ and $\varphi^3 = 1 + 2\varphi$,

$$\omega = \frac{R\varphi + P}{S\varphi + Q} = \frac{R\varphi^3 + P\varphi^2}{S\varphi^3 + Q\varphi^2} = \frac{R(1+2\varphi) + P(1+\varphi)}{S(1+2\varphi) + Q(1+\varphi)},$$

$$\omega = \frac{(2R+P)\varphi + R + P}{(2S+Q)\varphi + S + Q}. \tag{1}$$

Now, $(2R+P)(S+Q) - (2S+Q)(R+P) = -(PS - QR) = \pm 1$, and $S > -Q > 0$, so $2S + Q > S + Q > 0$.

In (1), $2R+P$, $2S+Q$, $R+P$, $S+Q$, and φ meet the conditions for Perron's P, Q, R, S, and ω in his Theorem 2.13. We conclude that in the RCF of ω, for an appropriate index r, $2R + P = A_r$, $R+P = A_{r-1}$, $2S+Q = B_r$, and $S+Q = B_{r-1}$. In turn, $R = A_r - A_{r-1}$, $P = 2A_{r-1} - A_r$, $S = B_r - B_{r-1}$, and $Q = 2B_{r-1} - B_r$.

We have $r \geq 1$ because $B_{r-1} = S + Q > 0$ and $B_{-1} = 0$.

Now we show that $a_r \geq 2$. First, $2S + Q > 2(S + Q) > 0$, so $B_r/B_{r-1} = (2S+Q)/(S+Q) > 2$ and $\lfloor B_r/B_{r-1} \rfloor \geq 2$. Also $B_i = a_i B_{i-1} + B_{i-2}$ for $i \geq 1$ [1, §5, p. 11], so $B_i/B_{i-1} = a_i + B_{i-2}/B_{i-1}$. Now, $B_{i-2} < B_{i-1}$ unless $i = 2$ and $a_1 = 1$. If $r \neq 2$ or $a_1 > 1$, then $\lfloor B_r/B_{r-1} \rfloor = a_r$, so $a_r \geq 2$. If $r = 2$ and $a_1 = 1$, then $B_1 = a_1 = 1 = S + Q$, so $B_2 = 2S + Q = S + 1$. Now, $S \geq 2$ (because $S > -Q > 0$), so $B_2 \geq 3$. As $B_2 = a_2 + 1$, it follows that $a_2 \geq 2$.

As ϕ is the associated complete quotient in $\omega = <a_0, \ldots, a_r, \varphi>$, and $\varphi = <\overline{1}>$, by [1, §8, Thm. 1.4, pp. 18–19] we have $\omega = <a_0, \ldots, a_r, \overline{1}>$.

□

References

[1] O. Perron, Die Lehre von den Kettenbrüchen, Band I: Elementare Kettenbrüche, Third Edition, B. G. Teubner Verlagsgesellschaft, Stuttgart, 1954.