MARKOV'S DOUBLY–INFINITE SEQUENCE OF FORMS

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Markov starts with an arbitrary indefinite binary quadratic form $f(x, y)$ and shows that the continued fraction algorithm provides a series of unimodular transformations (1) below (i.e., det = 1 or -1) that eventually convert $f(x, y)$ to what Gbur calls a Hermite reduced form. i.e. one with one root of $f(x, 1) = 0$ greater than 1, while the other is between -1 and 0.

So we start with the quadratic equation $a_0\xi^2 + b_0\xi + c_0 = 0$, where a_0, b_0, c_0 are integers, $d = b_0^2 - 4a_0c_0 > 0$ and not a perfect square. We assume that one root ξ is positive with $\xi_0 > 1$ and the other $-1/\eta_0$ is negative, with $-1 < -1/\eta_0 < 0$. Markov calls the form $a_0x_0^2+b_0x_0y_0+c_0y_0^2$ *reduced*. (M. Gbur calls the form *Hermite reduced*.) With Markov, we will use the term reduced in what follows. See [1] and [2].

Let

$$\xi_0 = [\alpha_0, \alpha_1, \ldots],$$

$$\eta_0 = [\alpha_{-1}, \alpha_{-2}, \ldots].$$

Markov defines a doubly–infinite sequence of forms $a_nx_n^2 + b_nx_ny_n + c_ny_n^2$ by means of the transformations

$$x_n = \alpha_n x_{n+1} + y_{n+1},$$

$$y_n = x_{n+1}, \quad n \in \mathbb{Z}. \tag{1}$$

Date: May 25, 2017.

Proposition 1. *We have recursion equations*

$$a_{n+1} = a_n\alpha_n^2 + b_n\alpha_n + c_n,$$

$$b_{n+1} = 2a_n\alpha_n + b_n,$$

$$c_{n+1} = a_n.$$

The form $a_nx_n^2 + b_nx_ny_n + c_ny_n^2$ is reduced, with positive and negative roots given by ξ_n and $-1/\eta_n$, where

$$\xi_n = [\alpha_n, \alpha_{n+1}, \ldots,],$$

$$\eta_n = [\alpha_{n-1}, \alpha_{n-2}, \ldots], \quad n \in \mathbb{Z}.$$

Corollary 2. *We have*

$$[\alpha_n, \alpha_{n+1}, \ldots,] + [0, \alpha_{n-1}, \alpha_{n-2}, \ldots] = \frac{\sqrt{d}}{|a_n|}. \tag{2}$$

Pavone [3] made important use of (2), with an explicit expression for a_n which was mentioned by Gbur and which was probably in Markov's master's thesis. Gbur (and doubtless Markov in his Master's thesis) defined two double–infinite sequences $(S_n), (T_n), n \in \mathbb{Z}$ by

$$S_0 = T_{-1} = 1, \quad S_{-1} = T_0 = 0$$

and $S_{k+1} = \alpha_kS_k + S_{k-1}, \quad T_{k+1} = \alpha_kT_k + T_{k-1}, \quad k \in \mathbb{Z}.$

Note that $S_{-k-1} > 0$ and $T_{-k-1} < 0$ precisely when $k \geq 1$ is odd.

Proposition 3. *For $n \in \mathbb{Z}$, we have*

$$a_n = a_0S_n^2 + b_0S_nT_n + c_oT_n^2,$$

$$b_n = 2a_0S_nS_{n-1} + b_0(S_nT_{n-1} + S_{n-1}T_n) + 2c_0T_nT_{n-1}.$$

Remark 4. These equations are mentioned indirectly in Delone [1].

Finally, we have a result which plays an important part in my paper and that of Pavone.

Corollary 5. *For $n \in \mathbb{Z}$, we have*

$$(3) \quad [\alpha_n, \alpha_{n+1}, \ldots,] + [0, \alpha_{n-1}, \alpha_{n-2}, \ldots] = \frac{\sqrt{d}}{|a_0 S_n^2 + b_0 S_n T_n + c_o T_n^2|}.$$

1. Examples

Example 6. The form $\Phi_0 = x_0^2 - 9x_0y_0 - y_0^2$ is Markov (Hermite) reduced. Here

$$\xi_0 = [\overline{9}], \quad \eta_0 = [\overline{9}]$$

and the Markov cycle is $\Phi_0 = (1, -9, -1), \Phi_1 = (-1, 9, 1)$.

Example 7. The form $\Phi_0 = 2x_0^2 - 4x_0y_0 - 3y_0^2$ is Markov (Hermite) reduced. Here

$$\xi_0 = [\overline{2, 1, 1}], \quad \eta_0 = [\overline{1, 1, 2}]$$

and the Markov cycle is

$$\Phi_0 = (2, -4, -3), \Phi_1 = (-3, 4, 2), \Phi_2 = (3, -2, -3), \Phi_3 = (-2, 4, 3), \Phi_4 = (3, -4, -2), \Phi_5 = (-3, 2, 3).$$

References

[1] B. N. Delone, *The St. Petersburg School of Number Theory*, History of Mathematics **26**, Amer. Math. Soc., 2005.

[2] A. A. Markov, Sur les formes binaires indefinies, *Math. Ann.* **15** (1879) 381-406.

[3] M. Pavone, A Remark on a Theorem of Serret, *J. Number Theory* **23** (1986) 268–278.