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A NOTE ON THE MARKOFF NUMBERS CONJECTURE

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Abstract. We show that the Markoff numbers conjecture is equivalent to a conjecture about the number of equivalence classes of a generalized Pell equation. The note originated in an attempt to understand papers by A. Baragar and J.O. Button, via the authors' background in the work of T. Nagell and B. Stolt. We give another proof of the unicity conjecture using the LMM algorithm, in the case $z = p^a$ or $2p^a$, where p is a prime of the form $4n + 1$.

1. Introduction

The diophantine equation

$$x^2 + y^2 + z^2 = 3xyz \tag{1.1}$$

was first studied by A. A. Markoff [6] in 1879. The positive solutions $(x, y, z), x \leq y \leq z$, of the diophantine equation form a tree rooted at $(1, 1, 1)$, using the branching operations

$$(x, y, z) \to (x, z, 3xz - y),$$

$$(x, y, z) \to (y, z, 3yz - x).$$

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See Cassells' book [4, p. 27–29]. Also see Zagier [12, p. 711–712] for the segment of the tree with $y \leq 100000$. The initial segment is given in Figure 1.

We refer to any positive solution (x, y, z) of (1.1) (not necessarily with $x \leq y \leq z$) as a *Markoff triple*. The numbers z in an ordered Markoff triple $x \leq y \leq z$ are called *Markoff numbers*.

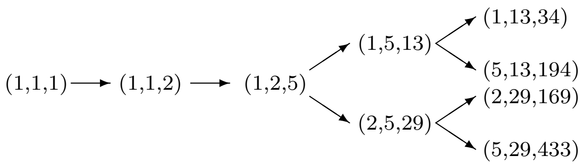

FIGURE 1. Initial portion of the Markoff tree.

It is a long–standing conjecture, first stated by Frobenius [5] in 1913, that if (x, y, z) and (x', y', z) are ordered Markoff triples, then $y = y'$ and $x = x'$. This is called the *Markoff numbers unicity conjecture*. The conjecture has been proved when z is an odd prime power, or twice an odd prime power, or when one of $3z-2$ and $3z+2$ is a prime power. See [9] for a list of references. The conjecture has been verified for $z \leq 10^{105}$ by Borosh [2]. We also have noticed that each Markoff number $z > 2$ appears exactly twice as the second component of an ordered Markoff triple. For example, 5 occurs only in the two ordered Markoff triples $1 < 5 < 13$ and $2 < 5 < 29$.

If (x, y, z) is a Markoff triple, we can rewrite (1.1) in the form

$$(3xz - 2y)^2 - (9z^2 - 4)x^2 = -4z^2$$

and we have an equation introduced by Baragar [1] and Button [3]:

$$X^2 - Dx^2 = -4z^2, \tag{1.2}$$

where $X = 3xz - 2y, D = 9z^2 - 4$. For future reference, we note that $X > 0$ if $y \leq z$. Conversely if (X, x) satisfies (1.2) with $z > 0, x > 0$, then $y = (3xz - X)/2$ is a positive integer and (x, y, z) is a Markoff triple. For $X^2 \equiv x^2z^2 \pmod 4$ and hence $y = (3xz - X)/2$ is an integer. Moreover $X^2 < 9x^2z^2$, so $|X| < 3xz$ and hence $y > 0$.

In this note, we relate the unicity conjecture to a conjecture about the number of equivalence classes of solutions of equation (1.2). Following Stolt [10], two solutions $(X, x), (X', x')$ of an equation

$$X^2 - Dx^2 = 4N \tag{1.3}$$

such as (1.2) are called *equivalent* if

$$X' + x'\sqrt{D} = (X + x\sqrt{D})(u + v\sqrt{D})/2,$$

where (u, v) satisfies the Pell equation $u^2 - Dv^2 = 4$. Classes come in pairs, with (X, x) and $(-X, x)$ in general defining different classes. If (X, x) and $(-X, x)$ define the same class, this class is called *ambiguous*. There are finitely many equivalence classes, each indexed by a *fundamental* solution (X_0, x_0) which has least positive x_0. In the case of an ambiguous class, we choose $X_0 \geq 0$. Stolt's definition of equivalence is related to that of Nagell [7], which is defined in terms of the Pell equation $u^2 - Dv^2 = 1$.

Our main result is:

Fundamental solution (X, x)	Markoff triple $(x, \frac{3xz-X}{2}, z)$
(1,1)	(1,1,1)
(4,1)	(1,1,2)
(11,1)	(1,2,5)
(29,1)	(1,5,13)
(164,2)	(2,5,29)
(76,1)	(1,13,34)
(199,1)	(1,34,89)
(956,2)	(2,29,169)
(2884,5)	(5,13,194)
(521,1)	(1,89,233)
(6437,5)	(5,29,433)
(1364,1)	(1,233,610)
(5572,2)	(2,169,985)
(51607,13)	(13,34,1325)
(3571,1)	(1,610,1597)
(43067,5)	(5,194,2897)
(9349,1)	(1,1597,4181)
(32476,2)	(2,985,5741)
(96124,5)	(5,433,6466)
(294491,13)	(13,194,7561)
(925676,34)	(34,89,9077)

TABLE 1. Markoff triples and corresponding fundamental solutions for $z \leq 10000$.

THEOREM 1.1. *Let $z \geq 5$. Then there is a 1–1 correspondence between the ordered Markoff triples (x, y, z) and the positive fundamental solutions of* (1.2).

This is a reformulation of A. Baragar's Theorem 1.1 of [1], which follows more directly by considering the orbits of an automorphism group which acts on the solutions of (1.1).

Table 1 lists the Markoff triples $(x, \frac{3xz-X}{2}, z)$ that arise from the positive fundamental solutions (X, x) of (1.2) over the range $1 \leq z \leq 10000$.

The paper depends on upper estimates of Stolt for the size of fundamental solutions. These estimates are similar to those of Nagell's who dealt with a definition of equivalence based on the Pell equation $u^2 - Dv^2 = 1$. In view of section 3 below, the unicity conjecture is equivalent to the statement that for $z \geq 5$, equation (1.2) has at most one solution (X, x) in positive integers, where $x \leq z/\sqrt{3z-2}$. This gives the unicity conjecture a similarity to the following two conjectures.

(i) The equation $x^2 - (k^2+1)y^2 = k^2, k > 1$ of Andrej Dujella [7] always has the solution $(x, y) = (k^2 - k + 1, k - 1)$ and is believed to have at most one other positive (exceptional) solution (x, y) with $1 \leq y < k - 1$. We get exceptional solutions if $k = 8, 12, 18, 21, 30, 32, 50, 55, \ldots$

(ii) The equation $x^2 - (n^2-1)y^2 = 2 - n^2$ of Kenji Kashihara [7] always has the solution $(x, y) = (1, 1)$ and appears to have at most one other positive solution (x, y) with $y \leq \sqrt{\frac{n^2-2}{2n-2}}$ unless $n = 33539$, when it has two, namely $x = 669941, y = 20$ and $x = 4326401, y = 129$. We get such solutions if $n = 11, 23, 39, 41, 59, 64, 83, 111, \ldots$ This sequence is listed on the OEIS site at `http://oeis.org/A130282`.

Interestingly, equation (1.2) and those of Dujella and Kashihara have a common form on rewriting:

$$9X^2 + 16 = (9z^2 - 4)(9y^2 - 4) \text{ (Markoff)}$$
$$x^2 + 1 = (k^2 + 1)(y^2 + 1) \text{ (Dujella)}$$
$$x^2 - 1 = (n^2 - 1)(y^2 - 1) \text{ (Kashihara)}.$$

These are all special cases of a general equation studied by Kashihara in [8].

2. Inequalities

Lemma 2.1. *Suppose $x \le y \le z$ is a Markoff triple. Then*

(i) $z > 2$ *implies* $x < y < z$;

(ii) $z \ge 5$ *implies* $x < \sqrt{z/3}$.

Proof. (i) See [4, p. 27].

(ii) We use induction on the tree. The inequality of (iii) is true for tree member $(1, 2, 5)$, so we assume its truth for tree member (x, y, z), where $z \ge 5$. We have to prove (a) $x < \sqrt{(3xz - y)/3}$ and (b) $y < \sqrt{(3yz - x)/3}$. For (a) we have

$$\begin{aligned} x < \sqrt{(3xz-y)/3} &\iff 3x^2 < 3xz - y, \\ &\iff 3x^2 y < 3xyz - y^2 = x^2 + z^2, \\ &\iff x^2(3y-1) < z^2. \end{aligned}$$

However the last inequality follows from Lemma 2.1 (iii):

$$x^2(3y-1) < (z/3)3y < (z/3)3z = z^2.$$

For (b) we have

$$\begin{aligned} y < \sqrt{(3yz-x)/3} &\iff 3y^2 < 3yz - x, \\ &\iff x < 3y(z-y) \end{aligned}$$

and the last inequality follows from $x < y < z$.

□

3. Fundamental solutions

Lemma 3.1. *Let* $u_1 + v_1\sqrt{D}$ *be the smallest positive solution of the Pell equation* $u_1^2 - Dv_1^2 = 4$. *Then a fundamental solution* $x + y\sqrt{D}$ *of the equation* $x^2 - Dy^2 = -4N, N > 0$ *must satisfy the inequalities:*

$$0 < y \leq \frac{v_1}{\sqrt{u_1 - 2}}\sqrt{N}, \tag{3.1}$$

$$0 \leq |x| \leq \sqrt{(u_1 - 2)N}. \tag{3.2}$$

The converse is easy to prove:

Lemma 3.2. *Suppose* $x^2 - Dy^2 = -4N$, *where* $N > 0$ *and*

$$0 < y \leq \frac{v_1}{\sqrt{(u_1 - 2)}}\sqrt{N}, \tag{3.3}$$

where $x \geq 0$ *in the case of equality. Then* $x + y\sqrt{D}$ *is a fundamental solution.*

Remark 3.3. It is straightforward to prove that fundamental solution $x + y\sqrt{D}$ defines an ambiguous class, if and only if equality holds in (3.3) or $x = 0$. In the case of equation (1.2), we have $u_1 = 3z, v_1 = 1$. Then $\frac{v_1}{\sqrt{(u_1-2)}}\sqrt{N} = z/\sqrt{3z-2}$ and this is an integer only when $z = 1$ and 2. Hence the only possibility for an ambiguous solution is $z = 1$ or 2. We find equation (1.2) has ambiguous solutions $(1, 1)$ or $(4, 1)$ if $z = 1$ or $z = 2$, respectively.

4. A one–to– one correspondence

In what follows, $z \geq 5$, $D = 9z^2 - 4$ and $\epsilon = (3z + \sqrt{D})/2$.

Lemma 4.1. *A positive solution* (X, x) *of* (1.2) *is a fundamental solution if and only if* $x \leq z/\sqrt{3z-2}$.

PROOF. This follows from Lemmas 3.1 and 3.2, as $x_1 = 3z$ and $y_1 = 1$. ∎

LEMMA 4.2. *If $x < y < z$ is a Markoff triple and $X = 3xz - 2y$, then (X, x) is a positive fundamental solution of* (1.2).

PROOF. We know (X, x) is a positive solution of (1.2). Also from Lemma 2.1, we have $x < \sqrt{z/3} < z/\sqrt{3z-2}$. Then Lemma 4.1 applies. ∎

LEMMA 4.3. *If (X, x) is a positive fundamental solution of* (1.2) *and $y = (3xz - X)/2$, then $x < y < z$ and (x, y, z) is a Markoff triple.*

PROOF. (i) We have $x < y$. For

$$\begin{aligned} x < y &\iff X < x(3z-2) \\ &\iff X^2 = (9z^2-4)x^2 - 4z^2 < x^2(3z-2)^2 \\ &\iff x^2(3z-2) < z^2. \end{aligned}$$

However the last inequality holds, as by Lemma 4.1,

$$x^2(3z-2) < (z/\sqrt{3z-2})^2(3z-2) = z^2.$$

(ii) We have $y < z$. For

$$\begin{aligned} y < z &\iff z(3x-2) < X \\ &\iff z^2(3x-2)^2 < (9z^2-4)x^2 - 4z^2 \\ &\iff 2z^2 - 3xz^2 + x^2 < 0. \end{aligned}$$

However $1 \le x < z/\sqrt{3z-2} < z$ and this implies $2z^2 - 3xz^2 + x^2 < 0$. ∎

Then Lemmas 4.2 and 4.3 imply that the ordered Markoff triples $x < y < z$ with z fixed, are in 1–1 correspondence with the positive fundamental solutions of generalized Pell equation (1.2).

5. Proof of the unicity conjecture when z is an odd prime power or twice an odd prime power

LEMMA 5.1. *Suppose* (X, x) *is a solution of* (1.2). *Then*

$$\gcd(X, x) = \begin{cases} 1 & \text{if } z \text{ is even, or } z \text{ is odd and } x \text{ is odd,} \\ 2 & \text{if } z \text{ is odd and } x \text{ is even.} \end{cases} \tag{5.1}$$

PROOF. Let $d = \gcd(X, x)$, where $X^2 - (9z^2 - 4)x^2 = -4z^2$. Then d^2 divides $4z^2$ and so d divides $2z$; also d divides x. But $\gcd(x, z) = 1$, for as was observed in the introduction, $y = (3xz - X)/2$ is an integer and x, y, z is a Markoff triple. Hence d divides 2. Hence if x is odd, $d = 1$; whereas if x is even, so is X and hence $d = 2$. □

We now prove the unicity conjecture holds when (a) $z = p^a$ or (b) $z = 2p^a$, where p is an odd prime of the form $4n + 1$, Let $x < y < z$ be a Markoff triple. Then with $X_1 = 3xz - 2y$, the equation $X_1^2 - Dx^2 = -4z^2$ is solvable in positive integers, where $\gcd(X_1, x) = 1$ or 2. We assume the reader is familiar with the LMM algorithm, which finds primitive fundamental solutions.

Case (a). If z is odd, we also have solutions $(X_2, y) = (3yz - 2x, y)$ and $(X_3, u) = (3uz - 2x, u)$, where $u = 3xz - y$. We see that exactly two of x, y, u are odd and one is even; without loss of generality take x and y odd, u even. It follows from Lemma 5.1 that (X_1, x) and (X_2, y) are primitive solutions of $X^2 - Dx^2 = -4z^2$, and can be verified are in distinct classes; whereas (X_3, u) is a primitive solution of $X^2 - Dx^2 =$

$-z^2$. But the congruence $P^2 \equiv D \pmod{4z^2}$ has 2 solutions in the range $0 < P < 2z^2$, while the congruence $P^2 \equiv D \pmod{z^2}$ has one solution in the range $0 < P < z^2$. Hence we know that there are respectively 4 and 2 primitive equivalence classes in the sense of Nagell equivalence. But with $\epsilon = (3z + \sqrt{D})/2$, we have relations

$$X_2 + y\sqrt{D} = (-X_1 + x\sqrt{D})\epsilon, \qquad X_3 + u\sqrt{D} = (X_1 + x\sqrt{D})\epsilon,$$
$$-X_2 + y\sqrt{D} = (X_1 + x\sqrt{D})\epsilon^{-1}, \quad -X_3 + u\sqrt{D} = (-X_1 + x\sqrt{D})\epsilon^{-1}.$$

Hence the solutions of $X^2-(9z^2-4)x^2 = -4z^2$ form two equivalence classes $(\pm X_1, x)$ in the sense of Stolt.

Case (b). Assume $z = 2p^a$. Then we prove that there are two equivalence classes in the sense of Stolt. We have to solve $X^2-(36p^{2a}-4)x^2 = -16p^{2a}$, so $X'^2 - dx^2 = p^{2a}$, where $X = 4X'$ and $d = (9p^{2a} - 1)/4$. But the congruence $P^2 \equiv d \pmod{p^{2a}}$ has but one solution in the range $0 < P < p^{2a}/2$, so we get just two primitive equivalence classes of solutions in the Nagell sense, with fundamental unit $3p^a + 2\sqrt{d} = (3z + \sqrt{D})/2 = \epsilon$. This results in two (primitive) equivalence classes of solutions of $X^2 - Dx^2 = -4z^2$ in the Stolt sense.

6. Examples

1. $z = 5$. Here $D = 221, 4z^2 = 100$ and the LMM algorithm gives Nagell primitive fundamental solutions $(\pm 11, 1)$ and $(\pm 193, 13)$ for $X^2 - 221x^2 = -100$, and gives Nagell primitive fundamental solutions $(\pm 14, 1)$ for $X^2 - 221x^2 = -25$. Hence we get two Stolt equivalence classes with fundamental solutions $(\pm 11, 1)$.

2. $z = 34$. Here $z = 2p, p = 17, D = 10400, d = 650$ and the LMM algorithm gives the Nagell primitive fundamental solutions $(\pm 19, 1)$ for $X'^2 - 650x^2 = -289$. Hence we get two Stolt equivalence classes with fundamental solutions $(\pm 76, 1)$ for $X^2 - 10400x^2 = -4624$.

7. Consequences of the unicity conjecture

We give a proof of a result of J. Button which describes the tree structure of the ordered Markoff triples which contain Markoff number z as a maximum element.

Lemma 7.1. *Suppose $a < b < z$ is a Markoff triple for which the Markoff conjecture holds. Let $X_0 = 3az - 2b$. Then the integer solutions (X, x) of $X^2 - Dy^2 = -4z^2$ are given by*

$$X + x\sqrt{D} = \pm(\pm X_0 + a\sqrt{D})\epsilon^n, n \in \mathbb{Z}. \tag{7.1}$$

Proof. This follows from the fact that the members of the equivalence class containing a solution (X_0, a) of (1.2) are given by

$$X + x\sqrt{D} = \pm(\pm X_0 + a\sqrt{D})\epsilon^n, n \in \mathbb{Z}.$$

∎

Lemma 7.2. *Suppose $a < b < z$ is an ordered Markoff triple and $X_0 = 3az - 2b$. Then for $n \in \mathbb{Z}$,*

$$(-X_0 + a\sqrt{D})\epsilon^n = u_n + v_n\sqrt{D}, \tag{7.2}$$

$$(X_0 + a\sqrt{D})\epsilon^{-n} = -u_n + v_n\sqrt{D}, \tag{7.3}$$

where $v_n > 0$ for $n \in \mathbb{Z}$, whereas $u_n > 0$ if $n \geq 1$ and $u_n < 0$ if $n \leq 0$.

Also for $n \in \mathbb{Z}$, we have recurrence relations

$$u_{n+1} = \frac{3zu_n + v_n D}{2}, \quad u_n = \frac{3zu_{n+1} - v_{n+1} D}{2}, \tag{7.4}$$

$$v_{n+1} = \frac{3zy_n + u_n}{2}, \quad y_n = \frac{3zy_{n+1} - u_{n+1}}{2}. \tag{7.5}$$

Finally, for $n \in \mathbb{Z}$, we have

$$v_{n+1} = 3zv_n - v_{n-1}, \tag{7.6}$$

where $v_0 = a$ and $v_1 = b$.

Proof. Equation (7.3) follows from (7.2) by conjugation. The recurrence relations follow from the equations

$$u_{n+1} + v_{n+1}\sqrt{D} = (u_n + v_n\sqrt{D})(3z + \sqrt{D})/2,$$
$$u_n + v_n\sqrt{D} = (u_{n+1} + v_{n+1}\sqrt{D})(3z - \sqrt{D})/2.$$

The other statements follow by induction on $n \geq 1$ using recurrence relations (7.4) and (7.5), noting that $u_1 = 3zb - 2a > 0$ and $v_1 = b > 0$ and by induction on $n \leq 0$, noting that $u_0 = -X < 0$ and $v_0 = a > 0$. ∎

Corollary 7.3. *Suppose $a < b < z$ is a Markoff triple for which the Markoff conjecture holds. Then all positive solutions (x, y, z) of the Markoff equation* (1.1) *are given by*

$$(v_n, v_{n+1}, z), (v_{n+1}, v_n, z), n \in \mathbb{Z},$$

where v_n are given by recurrence relation (7.6)

PROOF. The triples satisfy (1.1). For

$$\begin{aligned}
v_n^2 + v_{n+1}^2 + z^2 &= v_n^2 + (\frac{3zv_n + u_n}{2})^2 + z^2 \\
&= \frac{4v_n^2 + (9z^2v_n^2 + 6zu_nv_n + u_n^2) + 4z^2}{4} \\
&= \frac{18z^2v_n^2 + 6zu_nv_n}{4} \\
&= 3v_n(\frac{3zv_n + u_n}{2})z \\
&= 3v_nv_{n+1}z.
\end{aligned}$$

Conversely, suppose (x, y, z) is a Markoff triple. Then with $X = 3xz - 2y$, we have $X^2 - Dx^2 = -4z^2$. Hence from Lemma 7.1,

$$X + x\sqrt{D} = \pm(\pm X_0 + a\sqrt{D})\epsilon^n, n \in \mathbb{Z}. \tag{7.7}$$

where $X_0 = 3az - 2b$. It follows from Lemma 7.2 that the only possibilities in (7.7) for which the coefficient of $\sqrt{D}$ is positive are:

$$\text{(i) } X + x\sqrt{D} = (-X_0 + a\sqrt{D})\epsilon^n = u_n + v_n\sqrt{D},$$

$$\text{(ii) } X + x\sqrt{D} = (X_0 + a\sqrt{D})\epsilon^{-n} = -u_n + v_n\sqrt{D}.$$

There are two cases.

(i) $(X, x) = (-u_n, v_n)$. Then $3xz - 2y = -u_n, x = v_n$ and

$$y = \frac{3v_nz + u_n}{2} = v_{n+1}.$$

Hence $(x, y, z) = (v_n, v_{n+1}, z)$.

(ii) $(X, x) = (u_n, v_n)$. Then $3xz - 2y = u_n, x = v_n$ and from (7.5),

$$y = \frac{3v_nz - u_n}{2} = v_{n-1}.$$

Hence $(x, y, z) = (v_n, v_{n-1}, z)$.

list (7.8)	list (7.9)
(1,2,5)	(1,5,13)
(2,5,29)	(5,13,194)
(5,29,433)	(5,194,2897)
(5,433,6466)	(5,2897,43261)
(5,6466,96557)	(5,43261,646018)
(5,96557,1441889)	(5,646018,9647009)
(5,1441889,21531778)	(5,9647009,144059117)

FIGURE 2. Initial tree segment of triples containing $z = 5$.

∎

We then get the following result, which was pointed out in [3, p. 10–12], where there is a misprint on page 12, line −9, namely ${}^{z-1}z_0^c$ should be ${}^{c}z_0^{z-1}$.

COROLLARY 7.4. *Suppose $a < b < z$ is a Markoff triple for which the Markoff conjecture holds. Define the sequence $\{v_n\}$ for $n \in \mathbb{Z}$, by $v_0 = a, v_1 = b$ and $v_{n+2} = 3zv_{n+1} - v_n$ for $n \in \mathbb{Z}$. Then the ordered Markoff triples containing z are given by*

$$(v_0, v_1, z), (v_1, z, v_2) \text{ and } (z, v_n, v_{n+1}) \text{ for } n \geq 2; \tag{7.8}$$

$$(v_0, z, v_{-1}) \text{ and } (z, v_{-n+1}, v_{-n}) \text{ for } n \geq 2. \tag{7.9}$$

EXAMPLE 7.5. Taking $(a, b, z) = (1, 2, 5)$, Figure 2 lists the first seven ordered Markoff triples given by (7.8) and (7.9) which contain $z = 5$.

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