Computing the continued fraction of $\log_b a$

In 1954 Shanks gave an algorithm for computing the partial quotients of $\log_b a$, where a and b are integers, $a > b > 1$.

It is impractical to perform the above calculations in multiprecision arithmetic, as the process quickly grinds to a halt.

In this talk we outline a modification of Shanks' algorithm which produces partial quotients with a high degree of certainty.

Shanks' algorithm

Positive rationals $a_0, a_1, a_2, \ldots$ and positive integers $n_0, n_1, n_2, \ldots$ are constructed, as follows:

(i) $a_0 = a$ and $a_1 = b$;

(ii) If $i \geq 1$ and $a_{i-1} > a_i > 1$, then n_{i-1} and a_{i+1} are defined by

$$a_i^{n_{i-1}} \leq a_{i-1} < a_i^{n_{i-1}+1} \qquad (1)$$
$$a_{i+1} = a_{i-1}/a_i^{n_{i-1}}. \qquad (2)$$

Remarks

(i) (1) and (2) imply $a_i > a_{i+1} \geq 1$.

(ii) If $a_{i-1} > a_i > 1$, (1) implies

$$n_{i-1} = \left\lfloor \frac{\log a_{i-1}}{\log a_i} \right\rfloor. \qquad (3)$$

(ii) By induction on $j \geq 0$,

$$a_{2j} = a_0^r / a_1^s, \quad a_{2j+1} = a_1^u / a_0^v, \qquad (4)$$

where r and u are positive integers and s and v are non–negative integers.

Two possibilities:

(i) $a_{r+1} = 1$ for some $r \geq 1$.

Then (4) implies $a_0^q = a_1^p$ for positive integers p and q and so $\log_{a_1} a_0 = p/q$.

(ii) $a_{i+1} > 1$ for all i.

In this case the decreasing sequence $\{a_i\}$ tends to a limit $a \geq 1$. Also (1) implies $a_i \leq a_{i-1}^{1/n_{i-1}}$ and consequently

$$a_{i+1} \leq a_1^{1/n_1 \cdots n_i}.$$

Hence $a = 1$, unless $n_i = 1$ for all sufficiently large i.

But then (2) gives $a_{i+1} = a_{i-1}/a_i$ and hence $a = a/a = 1$.

If $a_{i+1} > 1$ we let $x_i = \log_{a_{i+1}} a_1$.

Lemma 1. If $a_{i+2} > 1$, then

$$x_i = n_i + \frac{1}{x_{i+1}}. \tag{4}$$

Proof. From equation (2), we have

$$\begin{aligned}
\log a_{i+2} &= \log a_i - n_i \log a_{i+1} \\
1 &= \frac{\log a_i}{\log a_{i+1}} \cdot \frac{\log a_{i+1}}{\log a_{i+2}} - n_i \cdot \frac{\log a_{i+1}}{\log a_{i+2}} \\
&= x_i x_{i+1} - n_i x_{i+1},
\end{aligned}$$

from which (4) follows.

From Lemma 1 and (3), we deduce

Lemma 2.

(a) If $\log_{a_1} a_0$ is irrational, then

$$x_i = n_i + \frac{1}{x_{i+1}} \text{ for all } i \geq 0.$$

(b) If $\log_{a_1} a_0$ is rational, with $a_{r+1} = 1$, then

$$x_i = \begin{cases} n_i + \frac{1}{x_{i+1}} & \text{if } 0 \leq i < r-1, \\ n_{r-1} & \text{if } i = r-1. \end{cases}$$

In view of the equation $x_0 = \log_{a_1} a_0$, Lemma 2 leads immediately to

Corollary.

$$\log_{a_1} a_0 = \begin{cases} [n_0, n_1, \ldots] & \text{if } \log_{a_1} a_0 \text{ is irrational,} \\ [n_0, n_1, \ldots, n_{r-1}] & \text{if } \log_{a_1} a_0 \text{ is rational} \\ & \text{and } a_{r+1} = 1. \end{cases}$$

Remark. It is an easy exercise to show that

$$a_{2j} = \frac{a_0^{q_{2j-2}}}{a_1^{p_{2j-2}}}, \quad a_{2j+1} = \frac{a_1^{p_{2j-1}}}{a_0^{q_{2j-1}}}$$

where p_k/q_k is the k–th convergent to $\log_{a_1} a_0$.

Example. $\log_2 10$: Here $a_0 = 10$, $a_1 = 2$.

(i) Then $2^3 < 10 < 2^4$, so $n_0 = 3$ and $a_2 = 10/2^3 = 1.25$.

(ii) Further, $1.25^3 < 2 < 1.25^4$, so $n_1 = 3$ and $a_3 = 2/1.25^3 = 1.024$.

(iii) Also, $1.024^9 < 1.25 < 1.024^{10}$, so $n_2 = 9$ and

$$\begin{aligned} a_4 &= 1.25/1.024^9 \\ &= \frac{1250000000000000000000000000}{1237940039285380274899124224} \\ &= 1.0097419586\cdots \end{aligned}$$

Continuing we obtain

i	n_i	a_i	p_i/q_i
0	3	10	3/1
1	3	2	10/3
2	9	1.25	93/28
3	2	1.024	196/59
4	2	$1.0097419586\cdots$	485/146
5	4	$1.0043362776\cdots$	2136/643
6	6	$1.0010415475\cdots$	13301/4004
7	2	$1.0001628941\cdots$	28738/8651
8	1	$1.0000637223\cdots$	42039/12655
9	1	$1.0000354408\cdots$	70777/21306
10		$1.0000282805\cdots$	
11		$1.0000071601\cdots$	

and $\log_2 10 = [3, 3, 9, 2, 2, 4, 6, 2, 1, 1, \ldots]$.

Pseudocode for the Shanks algorithm

```
Algorithm 0
input: integers a>b>1
output: partial quotients n[s] of log(a)/log(b)
        s:= 0
        a[0]=a; a[1]:=b
        aa:= a; bb:= b
        while(bb>1){
            i:=0
            while(aa>=bb){
                aa:= aa/bb
                i:= i+1
            }
            a[s+2]=aa
            n[s]=i
            t:= bb
            bb:= aa
            aa:= t
            s:= s+1
        }
```

In the exact–arithmetic language BC, if the *scale* is set to r digits, a positive rational number a is stored as $g(a)/10^r$, where $g(a) = \lfloor 10^r a \rfloor$. Eg. with $scale = 3$, $g(57/61) = 934$.

When dealing with positive rationals, BC calculates the quotient $a \oslash b$ using the following rule:

$$g(a \oslash b) = \lfloor 10^r g(a)/g(b) \rfloor. \qquad (1)$$

If we instead perform Algorithm 0 in BC, with $scale = r$ and with division `aa/bb` being interpreted as $aa \oslash bb$, the new algorithm will terminate and the integers `m[s]` will commence with partial quotients.

Naturally, the larger we take r, the more correct partial quotients will be expected.

The following pseudo code, when executed with $c = 10^r$, does the same job, working only with integers, using the g(a) values.

Pseudocode for the modified Shanks algorithm

```
Algorithm 1
input: integers a>b>1, c>1
output: positive integers m[s] which initially
        are partial quotients of log(a)/log(b)
        s:= 0
        A[0]=a*c; A[1]:=b*c
        aa:= A[0]; bb:= A[1]
        while(bb>c){
             i:=0
             while(aa>=bb){
                  aa:= floor(aa*c/bb)
                  i:= i+1
             }
             A[s+2]=aa
             m[s]=i
             t:= bb
             bb:= aa
             aa:= t
             s:= s+1
        }
```

Formal description of Algorithm 1

$A_0 = c \cdot a_0, A_1 = c \cdot a_1$.

If $i \geq 1$ and $A_{i-1} > A_i > c$, we define m_{i-1} and A_{i+1} by means of an intermediate sequence $\{B_{i,r}\}$, defined for $r \geq 0$, by $B_{i,0} = A_{i-1}$ and

$$B_{i,r+1} = \left\lfloor \frac{cB_{i,r}}{A_i} \right\rfloor, r \geq 0. \tag{5}$$

Then $c \leq B_{i,r+1} < B_{i,r}$, if $B_{i,r} \geq A_i$ and hence there is a unique integer $m = m_{i-1} \geq 1$ such that

$$B_{i,m} < A_i \leq B_{i,m-1}.$$

Then we define $A_{i+1} = B_{i,m}$.

Hence $A_{i+1} \geq c$ and the sequence $\{A_i\}$ decreases strictly until it reaches $A_{l(c)} = c$.

Theorem

(1) If $\log_{a_1} a_0 = p/q$, $p > q \geq 1$, $\gcd(p, q) = 1$, then

(a) $a_0 = d^p$, $a_1 = d^q$ for some positive integer d;

(b) if $p/q = [n_0, \ldots, n_{r-1}]$, where $n_{r-1} > 1$ if $r > 1$, then

(i) $A_{r+1} = c, a_{r+1} = 1$;

(ii) $A_i = c \cdot a_i$ for $0 \leq i \leq r + 1$;

(iii) $m_i = n_i$ for $0 \leq i \leq r - 1$.

(2) If $\log_{a_1} a_0$ is irrational, then

(a) $m_0 = n_0$;

(b) $l(c) \to \infty$ and for fixed i, $A_i/c \to a_i$ as $c \to \infty$ and $m_i = n_i$ for all large c.

Example $a_0 = 3, a_1 = 2$. Here are the $\{m_i\}$ for $\log_2 3$, with $c = 2^u, u = 1, \ldots, 31$:

```
1:  1,1,
2:  1,1,1,
3:  1,1,1,1,
4:  1,1,1,2,
5:  1,1,1,2,
6:  1,1,1,2,3,
7:  1,1,1,2,2,2,
8:  1,1,1,2,2,2,1,
9:  1,1,1,2,2,2,1,2,
10: 1,1,1,2,2,3,2,3,
11: 1,1,1,2,2,3,2,
12: 1,1,1,2,2,3,1,2, 1, 1,1, 2,
13: 1,1,1,2,2,3,1,3, 1, 1,3, 1,
14: 1,1,1,2,2,3,1,4, 3, 1,
15: 1,1,1,2,2,3,1,4, 1, 9,1,
16: 1,1,1,2,2,3,1,5,24, 1,2,
17: 1,1,1,2,2,3,1,5, 3, 1,1, 2,7,
18: 1,1,1,2,2,3,1,5, 2, 1,1, 5,3, 1,
19: 1,1,1,2,2,3,1,5, 2, 2,1, 3,1,16,
20: 1,1,1,2,2,3,1,5, 2,15,1, 6,2
21: 1,1,1,2,2,3,1,5, 2, 9,5, 1,2,
22: 1,1,1,2,2,3,1,5, 2,13,1, 1,1, 6,1,2,2,
23: 1,1,1,2,2,3,1,5, 2,17,2, 7,8,
24: 1,1,1,2,2,3,1,5, 2,19,1,49,2, 1,
25: 1,1,1,2,2,3,1,5, 2,22,4, 8,3, 4,1,
26: 1,1,1,2,2,3,1,5, 2,22,2, 1,3, 1,3, 8,
27: 1,1,1,2,2,3,1,5, 2,22,1, 6,3, 1,1, 3, 4,2,
28: 1,1,1,2,2,3,1,5, 2,23,2, 1,1, 2,1,12,17,
29: 1,1,1,2,2,3,1,5, 2,23,3, 2,2, 2,2, 1, 3,2,
30: 1,1,1,2,2,3,1,5, 2,23,2, 1,7, 2,2,14, 1,1,6,
31: 1,1,1,2,2,3,1,5, 2,23,2, 1,1, 1,1, 1, 8,2,1,14,2,
```

In fact
$\log_2 3 = [1, 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, \ldots]$

A heuristic algorithm

We can replace the $\lfloor x \rfloor$ function in equation (5) by $\lceil x \rceil$, the least integer exceeding x.

This produces an algorithm with similar properties to Algorithm 1, with integer sequences $\{A'_j\}$ and $\{m'_j\}$.

If we perform the two computations simultaneously, the common initial elements of the sequences $\{m_j\}$ and $\{m'_k\}$ are often partial quotients of $\log_b(a)$. Moreover, if $l(c) = l'(c)$ and $A_j = A'_j$ and $m_j = m'_j$ for $j = 0, \ldots, l(c)$, then $\log_b a$ is likely to be rational.

Example $c = 10^{10}$ and $a = 10, b = 2$,:

i	m_i	A_i	A'_i
0	3	100000000000	100000000000
1	3	20000000000	20000000000
2	9	12500000000	12500000000
3	2	10240000000	10240000000
4	2	10097419583	10097419591
5	4	10043362783	10043362769
6	6	10010415458	10010415496
7	2	10001629018	10001628856
8	1	10000636742	10000637759
9	1	10000355447	10000353253
10		10000281285	10000284496
11		10000074159	10000068756
12		10000058804	10000009470

Here $m_{10} = 3$ and $m'_{10} = 4$.

Example $a = 3, b = 2, c = 10^r, 1 \le r \le 20$:

```
 1: 1,1,1
 2: 1,1,1,2
 3: 1,1,1,2,2
 4: 1,1,1,2,2,3,1
 5: 1,1,1,2,2,3,1
 6: 1,1,1,2,2,3,1,5
 7: 1,1,1,2,2,3,1,5,2
 8: 1,1,1,2,2,3,1,5,2
 9: 1,1,1,2,2,3,1,5,2,23,2
10: 1,1,1,2,2,3,1,5,2,23,2,2
11: 1,1,1,2,2,3,1,5,2,23,2,2,1,1
12: 1,1,1,2,2,3,1,5,2,23,2,2,1,1
13: 1,1,1,2,2,3,1,5,2,23,2,2,1,1
14: 1,1,1,2,2,3,1,5,2,23,2,2,1,1
15: 1,1,1,2,2,3,1,5,2,23,2,2,1,1,55,1,4
16: 1,1,1,2,2,3,1,5,2,23,2,2,1,1,55,1,4
17: 1,1,1,2,2,3,1,5,2,23,2,2,1,1,55,1,4,3,1,1
18: 1,1,1,2,2,3,1,5,2,23,2,2,1,1,55,1,4,3,1,1
19: 1,1,1,2,2,3,1,5,2,23,2,2,1,1,55,1,4,3,1,1
20: 1,1,1,2,2,3,1,5,2,23,2,2,1,1,55,1,4,3,1,1,15,1
```

Example $a = 3, b = 2, c = 2^r, 1 \leq r \leq 31$: It seems likely that only partial quotients are produced for all $r \geq 1$.

```
 1: 1
 2: 1
 3: 1,1,1
 4: 1,1,1
 5: 1,1,1,2
 6: 1,1,1,2
 7: 1,1,1,2,2
 8: 1,1,1,2,2
 9: 1,1,1,2,2
10: 1,1,1,2,2
11: 1,1,1,2,2
12: 1,1,1,2,2
13: 1,1,1,2,2,3,1
14: 1,1,1,2,2,3,1
15: 1,1,1,2,2,3,1
16: 1,1,1,2,2,3,1,5
17: 1,1,1,2,2,3,1,5
18: 1,1,1,2,2,3,1,5
19: 1,1,1,2,2,3,1,5,2
20: 1,1,1,2,2,3,1,5
21: 1,1,1,2,2,3,1,5,2
22: 1,1,1,2,2,3,1,5,2
23: 1,1,1,2,2,3,1,5,2
24: 1,1,1,2,2,3,1,5,2
25: 1,1,1,2,2,3,1,5,2
26: 1,1,1,2,2,3,1,5,2
27: 1,1,1,2,2,3,1,5,2
28: 1,1,1,2,2,3,1,5,2,23
29: 1,1,1,2,2,3,1,5,2,23
30: 1,1,1,2,2,3,1,5,2,23,2
31: 1,1,1,2,2,3,1,5,2,23,2
```

Example $a = 34, b = 12, d = 10, r = 1, \ldots, 20.$

This example shows that we don't always get partial quotients.

```
test(34,12,10,1,20)
1: 1,2,2
2: 1,2,2,1,1
3: 1,2,2,1,1,2
4: 1,2,2,1,1,2
5: 1,2,2,1,1,2,3,1
6: 1,2,2,1,1,2,3,1,8,1
7: 1,2,2,1,1,2,3,1,8,1,1
8: 1,2,2,1,1,2,3,1,8,1,1,2
9: 1,2,2,1,1,2,3,1,8,1,1,2,2,1,13,3,2,32,7
10:1,2,2,1,1,2,3,1,8,1,1,2,2,1
11:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1
12:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1
13:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13
14:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,3
15:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2
16:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2
17:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2,18,1,1,1,1,1
18:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2,17,1
19:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2,17,1
20:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2,17,1
```

Properties of the integer part symbol give

$$\frac{G_{i-1,c}}{G_{i,c}^r} - \frac{(1-\frac{1}{G_{i,c}^r})}{c(1-\frac{1}{G_{i,c}})} < G_{i+1,c} \leq \frac{G_{i-1,c}}{G_{i,c}^r}.$$

This leads to the

Theorem With $G_{i,c} = A_{i,c}/c$ and $A_{i,c} > c + \sqrt{c}$, we have

$$\left\lfloor \frac{\log G_{i-1,c}}{\log G_{i,c}} \right\rfloor = m_{i-1,c} \text{ or } m_{i-1,c} + 1.$$

Similarly, properties of the ceiling function give

Theorem With $G'_{i,c} = A'_{i,c}/c$ and $A'_{i,c} > c + \sqrt{c}$, we have

$$\left\lfloor \frac{\log G'_{i-1,c}}{\log G'_{i,c}} \right\rfloor = m'_{i-1,c} \text{ or } m_{i-1,c} - 1,$$

if $c > 2$.

If $c = d^{2r}$, then provided $A_{i+1,c} \geq d^{2r} + d^{r+2}$, $m_{i-1,c}$ is likely to be a partial quotient of $\log_b a$.

One example where we do get an incorrect answer is $\log_{71}(74)$, when $d = 2, r = 10$: The sequence $m_{0,c}, m_{1,c}$, with $A_{3,c}$ the least $A_{i,c} \geq d^{2r} + d^{r+2}$, is $1, 102$, whereas $\log_{74}(71) = [1, 103, 15842, \ldots]$.

Contrastingly, with $a_0 = 3, a_1 = 2, d = 10, r = 250$, the same condition yields the correct first 482 partial quotients of $\log_2 3$.

The first 500 partial quotients of $\log_2 3$

1	1	1	2	2	3	1	5	2	23	2	2	1	1	55
1	4	3	1	1	15	1	9	2	5	7	1	1	4	8
1	11	1	20	2	1	10	1	4	1	1	1	1	1	37
4	55	1	1	49	1	1	1	4	1	3	2	3	3	1
5	16	2	3	1	1	1	1	1	5	2	1	2	8	7
1	1	2	1	1	3	3	1	1	1	1	5	4	2	2
2	16	8	10	1	25	2	1	1	1	2	18	10	1	1
1	1	9	1	5	6	2	1	1	12	1	1	1	6	2
12	1	1	12	1	1	2	12	1	12	3	1	5	1	14
1	1	14	2	3	1	2	2	1	4	1	4	8	1	1
1	3	5	1	1	1	1	2	1	4	3	7	5	3	1
32	1	1	1	18	1	3	2	5	2	1	3	1	8	1
1	1	2	6	6	5	33	2	2	3	1	1	1	1	29
1	3	2	1	21	1	6	52	1	8	1	4	14	9	7
1	4	18	2	2	1	1	2	100	39	1	2	1	1	19
1	5	9	1	3	964	5	1	1	1	39	1	1	1	1
5	3	1	88	1	2	1	3	1	11	1	23	11	1	1
1	2	1	1	4	3	1	5	1	4	2	1	75	1	2
1	11	17	2	5	3	1	3	34	1	10	2	4	7	1
1	23	1	6	3	1	7	1	17	2	1	24	1	1	1
10	1	4	1	1	5	3	2	1	2	1	1	3	6	8
1	8	2	1	1	4	2	7	9	2	2	2	1	7	12
2436	1	2	1	9	10	1	5	1	3	1	2	1	2	3
1	1	3	1	4	6	1	2	1	2	2	1	2	1	1
3	46	31	196	4	1	1	3	11	1	3	14	1	1	3
2	20	1	3	6	3	85	1	7	1	9	4	5	2	1
1	78	1	4	4	2	6	6	2	4	8	4	5	1	1
11	1	2	1	5	13	2	1	3	4	2	7	5	2	2
1	2	10	1	163	1	3	1	1	1	2	1	1	2	1
6	30	1	2	2	13	1	1	2	1	2	1	1	1	3
2	5	1	5	3	1	3	1	3	2	36	1	1	1	1
9	7	1	28	2	1	1	5	1	11	10	3	1	2	1
1	2	19	2	5	5	1	4	1	1	2	1	5	3	10
3	3	1	1	8										