INTERMEDIATE CONVERGENTS

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We are dealing with a regular continued fraction

$$\alpha = [a_0, a_1, \ldots].$$

The *n-th intermediate convergents* of $\alpha = [a_0, a_1, \ldots]$ are defined in terms of the convergents of α, as a ratio $p_{n,r}/q_{n,r}$, where

$$p_{n,r} = rp_{n+1} + p_n, \quad q_{n,r} = rq_{n+1} + q_n, \quad 1 \leq r < a_{n+2}, \quad n \geq -1.$$

In particular, when $r = a_{n+2} - 1$, we get the ratios $\frac{p_{n+2}-p_{n+1}}{q_{n+2}-q_{n+1}}, n \geq -1$. Note that $\frac{p_0-p_{-1}}{q_0-q_{-1}}$ is not defined to be an intermediate convergent. See Perron [2, p. 47] and Lang [1, pp. 15–18]. Perron gives initial values of $p_{n,r}/q_{n,r}$, together with convergents:

(a) n **even**:

$$\frac{p_0}{q_0}, \quad \frac{p_1+p_0}{q_1+q_0}, \quad \frac{2p_1+p_0}{2q_1+q_0}, \ldots, \frac{a_2p_1+p_0}{a_2q_1+q_0} = \frac{p_2}{q_2},$$
$$\frac{p_3+p_2}{q_3+q_2}, \quad \frac{2p_3+p_2}{2q_3+q_2}, \ldots, \frac{a_4p_3+p_2}{a_4q_3+q_2} = \frac{p_4}{q_4}, \ldots$$

(b) n **odd**:

$$\frac{p_0+p_{-1}}{q_0+q_{-1}}, \quad \frac{2p_0+p_{-1}}{2q_0+q_{-1}}, \ldots, \frac{a_1p_0+p_{-1}}{a_1q_0+q_{-1}} = \frac{p_1}{q_1},$$
$$\frac{p_2+p_1}{q_2+q_1}, \quad \frac{2p_2+p_1}{2q_2+q_1}, \ldots, \frac{a_3p_2+p_1}{a_3q_2+q_1} = \frac{p_3}{q_3}, \ldots$$

Theorem 1. (Lang [1, Thm. 9]) *For n even, we have a strictly increasing sequence of convergents and intermediate convergents less than*

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α

$$\cdots < \frac{p_n}{q_n} < \cdots < \frac{p_{n,r}}{q_{n,r}} < \frac{p_{n,r+1}}{q_{n,r+1}} < \cdots < \frac{p_{n+2}}{q_{n+2}} < \cdots$$

and a similar decreasing sequence for n odd and greater than α. Furthermore

$$q_{n,r+1}p_{n,r} - p_{n,r+1}q_{n,r} = (-1)^{n+1}, 0 \le r \le a_{n+2}.$$

Theorem 2. (Lang [1, Thm. 10]) *If p, q are non-zero integers, $q > 0$, satisfying $|\alpha - p/q| < 1/q^2$, then either p/q is a convergent of α or an intermediate convergent. In fact the latter has the form $(p_{n+1} + p_n)/(q_{n+1} + q_n), n \ge -1$ or $(p_{n+1} - p_n)/(q_{n+1} - q_n), n \ge 0$.*

Proof. This is a slightly expanded version of Lang's proof.

Case 1. Assume $\alpha < p/q$. Then

$$p/q < \alpha + 1/q^2 < a_0 + 1 + 1/q^2,$$
$$p < (a_0 + 1)q + 1,$$

so $p \le (a_0+1)q$ and $p/q \le (a_0+1)$. Then because the convergents and intermediate convergents with odd n partition the interval $(\alpha, a_0 + 1]$, there exist two consecutive members P/Q and P'/Q' such that p/q is not a convergent or intermediate convergent to α and

$$\alpha < P/Q < p/q < P'/Q',$$

where $P'Q - PQ' = 1$. Then

$$\frac{1}{q^2} > \frac{p}{q} - \alpha > \frac{p}{q} - \frac{P}{Q} \ge \frac{1}{qQ}$$

and

$$\frac{1}{Q'q} \le \frac{P'}{Q'} - \frac{p}{q} < \frac{P'}{Q'} - \frac{P}{Q} = \frac{1}{Q'Q}.$$

These estimates give a contradiction.

Case 2. Assume $\alpha > p/q$. Then

$$a_0 - \frac{p}{q} \leq \alpha - \frac{p}{q} < \frac{1}{q^2}$$
$$a_0 q - p < \frac{1}{q} \leq 1$$
$$a_0 q - p \leq 0.$$

Hence $a_0 \leq p/q$. Then because the convergents and intermediate convergents with even n partition the interval $[a_0, \alpha)$, there exist two consecutive members P/Q and P'/Q' such that p/q is not a convergent or intermediate convergent to α and

$$P'/Q' < p/q < P/Q < \alpha,$$

where $PQ' - QP' = 1$. Then

$$\frac{1}{q^2} > \alpha - \frac{p}{q} > \frac{P}{Q} - \frac{p}{q} \geq \frac{1}{qQ}$$

and

$$\frac{1}{Q'q} \leq -\frac{P'}{Q'} + \frac{p}{q} < -\frac{P'}{Q'} + \frac{P}{Q} = \frac{1}{Q'Q}.$$

These estimates give a contradiction. This finishes the first part of the proof. □

It remains to show that if p/q is not a convergent, then it has the form $p_{n,r}/q_{n,r}$, where $r = 1$ or $r = a_{n+2} - 1$. We need a lemma.

Lemma 3. *If $p_{n,r}/q_{n,r}$ is an intermediate convergent of α, then*

$$q_{n,r}\alpha - p_{n,r} = \frac{(-1)^n(\alpha_{n+2} - r)}{\alpha_{n+2} q_{n+1} + q_n}.$$

Proof. We use the equation

$$\alpha = \frac{p_{n+1}\alpha_{n+2} + p_n}{q_{n+1}\alpha_{n+2} + q_n} \tag{1}$$

to prove

$$q_n\alpha - p_n = \frac{(-1)^n \alpha_{n+2}}{\alpha_{n+2}q_{n+1} + q_n} \tag{2}$$

and

$$q_{n+1}\alpha - p_{n+1} = \frac{(-1)^n}{\alpha_{n+2}q_{n+1} + q_n}. \tag{3}$$

Multiplying (3) by r and adding (2) gives the assertion of the Lemma.

□

Back to the proof of Theorem 2. We observe that $r < \alpha_{n+2}$ as $r \le a_{n+2} - 1$. Hence

$$|q_{n,r}\alpha - p_{n,r}| = \frac{\alpha_{n+2} - r}{\alpha_{n+2}q_{n+1} + q_n}$$

and we have

$$\frac{\alpha_{n+2} - r}{\alpha_{n+2}q_{n+1} + q_n} < \frac{1}{rq_{n+1} + q_n}.$$

This inequality is equivalent to

$$\left(\frac{q_{n+1} + \frac{q_n}{r}}{q_{n+1} + \frac{q_n}{\alpha_{n+2}}}\right) r \left(1 - \frac{r}{\alpha_{n+2}}\right) < 1.$$

Suppose $r > 0$. Since $r < \alpha_{n+2}$, this inequality implies

$$r\left(1 - \frac{r}{\alpha_{n+2}}\right) < 1. \tag{4}$$

Suppose $r < a_{n+2} - 1$. Then $r \le a_{n+2} - 2 < \alpha_{n+2} - 2$. Hence

$$r\left(1 - \frac{\alpha_{n+2} - 2}{\alpha_{n+2}}\right) < 1,$$

and so $r < \alpha_{n+2}/2$. Hence from (4),

$$r\left(1 - \frac{\alpha_{n+2}}{2\alpha_{n+2}}\right) < 1,$$

and so $r < 2$. Hence $r = 1$. This finishes the proof of Theorem 2.

References

[1] S. Lang, *Introduction to Diophantine approximation*, Addison–Wesley Publishing Company, USA.

[2] O. Perron, *Die Lehre von den Kettenbrüchen*, Band 1, B.G. Teubner, Stuttgart, 1954.