Reduced quadratic irrationals and Pell's equation.

We give an account of Hasse's treatment of the connection between reduced quadratic irrationals and the fundamental solution of Pell's equation from *Vorlesung über Zahlentheorie.*

$D > 0$ is not a perfect square, $D \equiv 0$ or $1) \pmod 4$.

θ and θ' are the roots of

$$ax^2 - bx + c = 0, \tag{1}$$

where $d = b^2 - 4ac$, $a > 0$ and $\gcd(a, b, c) = 1$.

$$\theta = \frac{b + \sqrt{d}}{2a}, \quad \theta' = \frac{b - \sqrt{d}}{2a}.$$

θ is called *reduced* if $1 < \theta$ and $-1 < \theta' < 0$. Equivalently

$$0 < b < \sqrt{d}, \quad 2a - b < \sqrt{d} < 2a + b.$$

Note: $c > 0$.

If θ is reduced, then $\theta = [\overline{a_0, \ldots, a_k}]$.

Consequently

$$\theta = \frac{p_k\theta + p_{k-1}}{q_k\theta + q_{k-1}}, \tag{2}$$

where $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1}$.

Equation (2) implies

$$q_k\theta^2 - (p_k - q_{k-1})\theta - p_{k-1} = 0. \tag{3}$$

Equation (2) also implies the existence of ϵ such that

$$\epsilon \begin{pmatrix} \theta \\ 1 \end{pmatrix} = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix}. \tag{4}$$

Hence ϵ is an eigenvalue of $\begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}$ and

$$\epsilon^2 - (p_k + q_{k-1})\epsilon + (-1)^{k+1} = 0. \tag{5}$$

Equation (4) implies

$$\epsilon = q_k\theta + q_{k-1}. \tag{6}$$

Let $v = \gcd(q_k, p_k - q_{k-1}, p_{k-1})$.

Then comparing equations (1) and (3) gives

$$q_k = av, \; p_k - q_{k-1} = bv, \; p_{k-1} = -cv. \tag{7}$$

Now let $p_k + q_{k-1} = u$. Then (6) gives

$$\begin{aligned} \epsilon &= (av)\theta + \frac{u - bv}{2} \\ &= v\left(\frac{b + \sqrt{d}}{2}\right) + \frac{u - bv}{2} \\ &= \frac{u + v\sqrt{d}}{2}. \end{aligned}$$

Also

$$\begin{aligned} (2\epsilon - u)^2 - v^2 d &= 0 \qquad (8) \\ \epsilon^2 - u\epsilon + \frac{u^2 - v^2 d}{4} &= 0. \end{aligned}$$

Comparing (5) and (9) gives

$$\frac{u^2 - v^2 d}{4} = (-1)^{k+1}. \qquad (9)$$

Note: $u \geq 1, v \geq 1$.

Now assume that $x^2 - dy^2 = \pm 4$, with $x \geq 1$ and $y \geq 1$. Then we prove $\eta = (x + y\sqrt{d})/2 = \epsilon^t$ for some $t \geq 1$. This characterises ϵ as the smallest solution of $x^2 - dy^2 = \pm 4$.

Let

$$p = \frac{x + by}{2},\ p' = -cy,\ q = ay,\ q' = \frac{x - by}{2}.$$

Then

$$pq' - p'q = \frac{x^2 - dy^2}{4} = \pm 1. \qquad (10)$$

Also

$$\begin{aligned} q\theta^2 - (p - q')\theta - p' &= ay\theta^2 - by\theta + cy \\ &= y(a\theta^2 - b\theta + c) = 0. \end{aligned}$$

Hence

$$\theta = \frac{p\theta + p'}{q\theta + q'}. \qquad (11)$$

Hasse then proves (see later)

$$\begin{cases} q \geq q' > 0 & \text{if } \frac{x^2 - dy^2}{4} = 1, \\ q > q' \geq 0 & \text{if } \frac{x^2 - dy^2}{4} = -1. \end{cases} \qquad (12)$$

It follows from Theorem 172 of Hardy and Wright, that $p/q = p_n/q_n, p'/q' = p_{n-1}/q_{n-1}$, θ is the $(n+1)$-th complete quotient in the cfrac of θ and that

$$\theta = \frac{p_n\theta + p_{n-1}}{q_n\theta + q_{n-1}}. \qquad (13)$$

It follows that $n + 1$ is a multiple of the period k of the cfrac for θ, $n + 1 = t(k + 1)$ and that $\eta = \epsilon^t$. This is standard, but we prove it.

First note that

$$\begin{aligned} \eta\theta &= p_n\theta + p_{n-1} \\ \eta &= q_n\theta + q_{n-1}. \end{aligned}$$

Iterating (4) t times gives

$$\begin{aligned} \epsilon^t \begin{pmatrix} \theta \\ 1 \end{pmatrix} &= \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}^t \begin{pmatrix} \theta \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix} = \eta \begin{pmatrix} \theta \\ 1 \end{pmatrix}. \end{aligned}$$

Hence $\eta = \epsilon^t$.

Finally, we remark that if $x_0, \ldots, x_k$ are the complete quotients of $x_0 = \theta$, then

$$x_0 \cdots x_k = \epsilon. \tag{14}$$

This follows from a result of H.J.S. Smith:

$$x_0 \cdots x_k = \frac{(-1)^{k+1}}{p_k - q_k\theta}. \tag{15}$$

Now multiply the numerator and denominator of the RHS of (15) by $p_k - q_k\theta'$.

The denominator simplifies to $(-1)^{k+1}$, while the numerator becomes $(u + v\sqrt{d})/2 = \epsilon$. (Details omitted).

Regarding Theorem 172, Hasse needs a slight extension of it - one case being mentioned in my Bordeaux paper, namely if $S = 0$ and $Q = R = 1$. The other is if $Q = S$ and $P = R+1$. These cases are relevant when equality occurs in cases (12) respectively.

Hasse's Proof. The reduced nature of θ means

$$0 < b < \sqrt{d},\ 2a - b < \sqrt{d} < 2a + b.$$

We also note that $\eta = (x + y\sqrt{d})/2 > 1$. Also

$$q' = \frac{x - by}{2} > \frac{x - y\sqrt{d}}{2} = \epsilon' = \frac{N(\epsilon)}{\epsilon} > \begin{cases} 0 & \text{if } N(\epsilon) = 1, \\ -1 & \text{if } N(\epsilon) = -1. \end{cases}$$

Next

$$q - q' = \frac{-x + (2a + b)y}{2} > \frac{-x + y\sqrt{d}}{2} = -\epsilon' = -\frac{N(\epsilon)}{\epsilon} > \begin{cases} -1 & \text{if } N(\epsilon) = 1, \\ 0 & \text{if } N(\epsilon) = -1. \end{cases}$$

Hence

$$\begin{aligned} 0 < q' \le q \quad &\text{if} \quad N(\epsilon) = 1, \\ 0 \le q' < q \quad &\text{if} \quad N(\epsilon) = -1. \end{aligned}$$