An inequality for the determinant of a positive definite Hermitian matrix

The following result was obtained by Professor Grainger Morris, formerly of the University of New England. Grainger asked for a simple proof and I came up with the following on 18th November 1996.

THEOREM If $H = A + iB$ is a positive–definite Hermitian matrix, A, B real, then $\det H \leq \det A$.

PROOF Let $H = A + iB$ and $D = \left[\begin{array}{c|c} A & -B \\ \hline B & A \end{array}\right]$. Then

1. H is Hermitian if and only if A is symmetric and B is skew–symmetric.
2. H is Hermitian if and only if D is symmetric.
3. H is Hermitian positive–definite if and only if D is symmetric positive–definite.
4. $\det D = (\det H)^2$. In fact $ch_D(x) = (ch_H(x))^2$. (Use elementary row and column operations.)
5. If D is symmetric positive definite, then by Fischer's inequality (see L. Mirsky, *An Introduction to Linear Algebra*, OUP 1961, Theorem 13.5.5, page 420), we have
$$\det D \leq (\det A)^2.$$

Then the desired inequality $\det H \leq \det A$ follows from (3) and (4).