The fact that the decimal expansion of a rational number is eventually periodic is a consequence of the pigeonhole principle.

We start by examining the process of *long–division*. To find the decimal expansion of $1/7$ we perform the following steps:

$$
\begin{array}{l}
\quad\;\underline{\cdot 1428571\cdots} \\
7\sqrt{10} \\
\quad\;\underline{7} \\
\quad\;\;30 \\
\quad\;\;\underline{28} \\
\quad\;\;\;\;20 \\
\quad\;\;\;\;\underline{14} \\
\quad\;\;\;\;\;\;60 \\
\quad\;\;\;\;\;\;\underline{56} \\
\quad\;\;\;\;\;\;\;\;40 \\
\quad\;\;\;\;\;\;\;\;\underline{35} \\
\quad\;\;\;\;\;\;\;\;\;\;50 \\
\quad\;\;\;\;\;\;\;\;\;\;\underline{49} \\
\quad\;\;\;\;\;\;\;\;\;\;\;\;10 \\
\quad\;\;\;\;\;\;\;\;\;\;\;\;\;\;\cdots
\end{array}
$$

We have thus performed the following calculations:

$10 \cdot 1 = 10 = 7 \cdot 1 + 3$

$10 \cdot 3 = 30 = 7 \cdot 4 + 2$

$10 \cdot 2 = 20 = 7 \cdot 2 + 6$

$10 \cdot 6 = 60 = 7 \cdot 8 + 4$

$10 \cdot 4 = 40 = 7 \cdot 5 + 5$

$10 \cdot 5 = 50 = 7 \cdot 7 + 1.$

After this the previous steps are replicated.

More generally, to find the decimal expansion of a/b, where $1 \leq a < b$ and $b > 1$, we let $r_0 = a$ and perform the steps;

$$10 \cdot r_0 = b \cdot a_1 + r_1, \quad 0 \leq r_1 < b,$$

$$10 \cdot r_1 = b \cdot a_2 + r_2, \quad 0 \leq r_2 < b,$$

$$\vdots$$

The $a_1, a_2, \ldots$ are the digits of the decimal expansion of a/b. The remainders $r_0, r_1, \ldots$ all lie in the range $0 \leq r_i \leq b - 1$. So by applying the pigeonhole principle to the numbers $r_0, \ldots, r_b$, we will get $r_i = r_j$ for some i and j satisfying $0 \leq i < j \leq b$. Then

$a_{i+1} = a_{j+1}, r_{i+1} = r_{j+1}, a_{i+2} = a_{j+2}, \ldots$ and the decimal expansion is periodic.

EXAMPLES. $\frac{7}{12} = \cdot 58\dot{3}$; $\frac{3}{14} = \cdot 2\dot{1}4285\dot{7}$.