CONTINUANTS AND SEMI-REGULAR CONTINUED FRACTIONS

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Abstract. This note arose while studying Perron's proof of Satz 5.1, [2, p. 135]. Perron used inequalities involving the $B_{\nu,\lambda}$ and Keith Matthews challenged the author to supply a proof using only the B_λ. This resulted in a simpler proof of Lemma 6 below. A self-contained treatment of the continuants $A_{\nu,\lambda}$ and $B_{\nu,\lambda}$ is given for the convenience of the reader.

Let $A_{-1} = 1, A_0 = b_0, B_{-1} = 0, B_0 = 1$ and for $n \geq 1$,

$$A_n = b_n A_{n-1} + a_n A_{n-2}, \tag{1}$$

$$B_n = b_n B_{n-1} + a_n B_{n-2}. \tag{2}$$

Lemma 1. *If* $|a_n| = 1, b_n \geq 1, b_n + a_{n+1} \geq 1$, $n \geq 1$, *then for* $n \geq 0$,

(a) $B_n \geq 1$,
(b) $B_n + a_{n+1} B_{n-1} \geq 1$.

Proof. (Induction on $n \geq 0$.) (a) and (b) are true when $n = 0$, as $B_0 = 1$ and $B_0 + a_1 B_{-1} = B_0 = 1$. Now assume (a) and b) hold for all $k \leq n$. Then

$$\begin{aligned} B_{n+1} + a_{n+2} B_n &= (b_{n+1} B_n + a_{n+1} B_{n-1}) + a_{n+2} B_n \\ &= B_n (b_{n+1} + a_{n+2}) + a_{n+1} B_{n-1} \\ &\geq B_n + a_{n+1} B_{n-1} \geq 1. \end{aligned}$$

Also

$$\begin{aligned} B_{n+1} &= b_{n+1} B_n + a_{n+1} B_{n-1} \\ &\geq B_n + a_{n+1} B_{n-1} \\ &= (b_n B_{n-1} + + a_n B_{n-2}) + a_{n+1} B_{n-1} \\ &= (b_n + a_{n+1}) B_{n-1} + a_n B_{n-2} \\ &\geq B_{n-1} + a_n B_{n-2} \geq 1. \end{aligned}$$

□

Remark. It follows from the above proof that

$$B_n \geq B_{n-1} + a_n B_{n-2} \geq B_{n-2} + a_{n-1} B_{n-3} \geq \cdots B_0 + a_1 B_{-1} = 1. \tag{3}$$

Hence, with a suitable recursive definition of the RHS below (see `http://www.numbertheory.org/courses/MP313/lectures/lecture15/page5.html`) we have

Lemma 2.

$$\frac{A_\nu}{B_\nu} = b_0 + \frac{a_1|}{|b_1} + \cdots + \frac{a_n|}{|b_n}. \tag{4}$$

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The following result is due to Fritz Blumer [1, p. 12].

Lemma 3. *If* $|a_n| = 1, b_n \geq 1, b_n + a_{n+1} \geq 1,\ n \geq 1,$ *then for* $n \geq 1$,

$$b_n \geq 2 \implies B_n > B_{n-1}. \tag{5}$$

Proof. Suppose $b_n \geq 2, n \geq 1$. Then

$$\begin{aligned} B_n &= b_n B_{n-1} + a_n B_{n-2} \\ &\geq 2B_{n-1} + a_n B_{n-2} \\ &= B_{n-1} + (B_{n-1} + a_n B_{n-2}) \\ &\geq B_{n-1} + 1 \text{ from Lemma 1 (b).} \end{aligned}$$

□

Perron ([2, p. 10]) defines $A_{\nu,\lambda}$ and $B_{\nu,\lambda}$ in terms of *continuants*, as follows:

$$A_{\nu,\lambda} = K\begin{pmatrix} a_{\lambda+1}, a_{\lambda+2}, \ldots, a_{\lambda+\nu} \\ b_\lambda, b_{\lambda+1}, \ldots\ldots, b_{\lambda+\nu} \end{pmatrix}, \nu \geq 1, \tag{6}$$

$$B_{\nu,\lambda} = K\begin{pmatrix} a_{\lambda+2}, a_{\lambda+3}, \ldots, a_{\lambda+\nu} \\ b_{\lambda+1}, b_{\lambda+2}, \ldots\ldots, b_{\lambda+\nu} \end{pmatrix}, \nu \geq 2, \tag{7}$$

where $A_{-1,\lambda} = 1, B_{-1,\lambda} = 0, A_{0,\lambda} = b_\lambda, B_{0,\lambda} = 1, B_{1,\lambda} = b_{\lambda+1}$.

Here

$$K\begin{pmatrix} a_1, a_2, \ldots, a_\nu \\ b_0, b_1, b_2, \ldots, b_\nu \end{pmatrix} = \begin{vmatrix} b_0 & -1 & 0 & \cdots & & & \\ a_1 & b_1 & -1 & 0 & \cdots & & \\ 0 & a_2 & b_2 & -1 & 0 & \cdots & \\ & & & \cdots & & & \\ & & & \cdots & a_{\nu-1} & b_{\nu-1} & -1 \\ & & & \cdots & 0 & a_\nu & b_\nu \end{vmatrix}. \tag{8}$$

Thus $A_{\nu,0} = A_\nu, B_{\nu,0} = B_\nu$. Also From (6) and (7), we have

$$B_{\nu,\lambda} = A_{\nu-1,\lambda+1}. \tag{9}$$

Expanding the determinant in (8) gives a recurrence relation connecting continuants:

Lemma 4. *For* $\nu \geq 3$,

$$K\begin{pmatrix} a_1, a_2, \ldots, a_\nu \\ b_0, b_1, b_2, \ldots, b_\nu \end{pmatrix} = b_0 K\begin{pmatrix} a_2, a_2, \ldots, a_\nu \\ b_1, b_1, b_2, \ldots, b_\nu \end{pmatrix} + a_1 K\begin{pmatrix} a_3, a_2, \ldots, a_\nu \\ b_2, b_1, b_2, \ldots, b_\nu \end{pmatrix}.$$

From Lemma 4, we deduce from (6) and (7) that for $\nu \geq 3, \lambda \geq 0$,

$$A_{\nu,\lambda} = b_\lambda A_{\nu-1,\lambda+1} + a_{\lambda+1} A_{\nu-2,\lambda+2}, \tag{10}$$

$$B_{\nu,\lambda} = b_{\lambda+1} B_{\nu-1,\lambda+1} + a_{\lambda+2} B_{\nu-2,\lambda+2}. \tag{11}$$

In fact, equations (10) and (11) hold for $\nu \geq 1$.

Corollary 1. *If* $0 \leq \lambda \leq n$, *then*

$$A_n = A_{\lambda-1} A_{n-\lambda,\lambda} + a_\lambda A_{\lambda-2} B_{n-\lambda,\lambda}, \tag{12}$$

$$B_n = B_{\lambda-1} A_{n-\lambda,\lambda} + a_\lambda B_{\lambda-2} B_{n-\lambda,\lambda}. \tag{13}$$

Proof. (By conservation). Let $f(\lambda) = A_{\lambda-1}A_{n-\lambda,\lambda} + a_\lambda A_{\lambda-2}B_{n-\lambda,\lambda}$.
Then $f(0) = A_{-1}A_{n,0} + a_0 A_{-2}B_{n,0} = A_n$.
Next we prove $f(\lambda) = f(\lambda+1)$ for $0 \le \lambda \le n$.

$$\begin{aligned} f(\lambda+1) &= A_\lambda A_{n-\lambda-1,\lambda+1} + a_{\lambda+1}A_{\lambda-1}B_{n-\lambda-1,\lambda+1} \\ &= (b_\lambda A_{\lambda-1} + a_\lambda A_{\lambda-2})A_{n-\lambda-1,\lambda+1} + a_{\lambda+1}A_{\lambda-1}B_{n-\lambda-1,\lambda+1} \\ &= A_{\lambda-1}(b_\lambda A_{n-\lambda-1,\lambda+1} + a_{\lambda+1}B_{n-\lambda-1,\lambda+1}) + a_\lambda A_{\lambda-2}A_{n-\lambda-1,\lambda+1} \\ &= A_{\lambda-1}(b_\lambda A_{n-\lambda-1,\lambda+1} + a_{\lambda+1}A_{n-\lambda-2,\lambda+2}) + a_\lambda A_{\lambda-2}B_{n-\lambda,\lambda} \\ &= A_{\lambda-1}A_{n-\lambda,\lambda} + a_\lambda A_{\lambda-2}B_{n-\lambda,\lambda} = f(\lambda). \end{aligned}$$

Hence $A_n = f(0) = f(1) = \cdots = f(n)$. □

Lemma 5. *If* $|a_n| = 1, b_n \ge 1, b_n + a_{n+1} \ge 1$, $n \ge 1$, *then for* $1 \le \mu \le \lambda$, ,

$$B_{\mu,\lambda-\mu} \ge B_{\mu-1,\lambda-\mu+1}. \tag{14}$$

Proof. ([2, p. 135].) From (11), we have

$$B_{\nu+1,\lambda-\nu-1} - B_{\nu,\lambda-\nu} = (b_{\lambda-\nu} - 1)B_{\nu,\lambda-\nu} + a_{\lambda-\nu+1}B_{\nu-1,\lambda-\nu+1}. \tag{15}$$

We prove (14) by induction on μ. First $B_{1,\lambda-1} = b_\lambda, B_{0,\lambda} = 1$. So we assume (14) holds for $\mu = 1, \dots, \nu < \lambda$, Then

$$B_{\nu,\lambda-\nu} \ge B_{\nu-1,\lambda-\nu+1} \ge \cdots \ge B_{0,\lambda} = 1.$$

Then (15) implies

$$B_{\nu+1,\lambda-\nu-1} - B_{\nu,\lambda-\nu} \ge (b_{\lambda-\nu} - 1 + a_{\lambda-\nu+1})B_{\nu-1,\lambda-\nu+1} \ge 0.$$

Hence the induction goes through. □

Corollary 2.

$$B_\lambda = B_{\lambda,0} \ge B_{\lambda-1,1} \ge B_{\lambda-2,2} \ge \cdots \ge B_{0,\lambda} = 1. \tag{16}$$

Hence, for $0 \le \nu \le n$,

$$\frac{A_{\nu,n-\nu}}{B_{\nu,n-\nu}} = b_{n-\nu} + \frac{a_{n-\nu+1}|}{|b_{n-\nu+1}} + \cdots + \frac{a_n|}{|b_n}. \tag{17}$$

Perron (Satz 5.1, [2, p. 135]) proved the following results using continuants.

Theorem 1. (Tietze, [3]). *Suppose* $a_\lambda = \pm 1, b_\lambda \ge 1$ *and* $b_\lambda + a_{\lambda+1} \ge 1$ *for* $\lambda \ge 1$. *Then*

(i) $B_\lambda \to \infty$,
(ii) A_λ/B_λ *converges as* $\lambda \to \infty$.

Remark. Perron proved (i) by showing

Lemma 6. (a) $a_{\lambda+1} = 1 \implies B_{\lambda+\nu} \ge \lambda + 1$,
(b) $a_{\lambda+1} = -1 \implies B_\lambda \ge \lambda + 1$.

Proof. (Alan Offer) We use induction on $\lambda \ge 0$. When $\lambda = 0$, (a) is true, as Lemma 1 (a) implies $B_\nu \ge 1$ for $\nu \ge 1$. Also (b) is true as $B_0 = 1$. So let $\lambda > 0$ and suppose that (a) and (b) hold for all $\lambda' < \lambda$. We first show that (a) holds. Suppose $a_{\lambda+1} = 1$. Then for all $\nu \ge 1$, (3) implies

$$B_{\lambda+\nu} \ge B_\lambda + B_{\lambda-1}. \tag{18}$$

Now if $a_\lambda = 1$, then $B_\lambda \geq \lambda$ by (a), while if $a_\lambda = -1$, then $B_{\lambda-1} \geq \lambda$ by (b). Either way, (18) implies $B_{\lambda+\nu} \geq \lambda + 1$ if $\nu \geq 1$. This completes the inductive step for (a).

For (b), we suppose $a_{\lambda+1} = -1$. Then $b_\lambda \geq 2$ and Lemma 3 gives $B_\lambda > B_{\lambda-1}$.

If $a_\lambda = -1$, (b) implies $B_{\lambda-1} \geq \lambda$ and hence $B_\lambda \geq \lambda + 1$.

Now assume $a_\lambda = 1$. Then $B_\lambda \geq 2B_{\lambda-1} + B_{\lambda-2}$. If $\lambda = 1$, then

$$B_\lambda = B_1 \geq 2B_0 + B_{-1} = 2 = \lambda + 1.$$

So we can assume $\lambda \geq 2$.

If $a_{\lambda-1} = -1$, then (b) implies $B_{\lambda-2} \geq \lambda - 1$ and so

$$B_\lambda \geq 2B_{\lambda-1} + \lambda - 1 \geq \lambda + 1.$$

If $a_{\lambda-1} = 1$, then $\lambda \geq 2$ implies $B_{\lambda-2} \geq 1$. Also (a) with λ replaced by $\lambda - 2$ and $\nu = 1$, gives $B_{\lambda-1} \geq \lambda - 1$. Hence

$$\begin{aligned} B_\lambda &\geq 2B_{\lambda-1} + B_{\lambda-2} \\ &\geq 2(\lambda - 1) + \lambda - 1 = 2\lambda - 1 \geq \lambda + 1. \end{aligned}$$

□

Finally, (ii) follows from (i) and the inequality

$$\left| \frac{A_{\lambda+\nu}}{B_{\lambda+\nu}} - \frac{A_\lambda}{B_\lambda} \right| \leq \frac{1}{B_\lambda}, \tag{19}$$

which is proved below.

Lemma 7. *For $\lambda \geq 0, \nu \geq 1$, let*

$$D_{\lambda,\nu} = A_{\lambda+\nu} B_\lambda - A_\lambda B_{\lambda+\nu}. \tag{20}$$

Then

$$D_{\lambda,\nu} = (-1)^\lambda a_1 a_2 \cdots a_{\lambda+1} B_{\nu-1,\lambda+1}, \tag{21}$$

Proof. Perron ([2, p. 14]) derives (21) from (12) and (13).

$$\begin{aligned} D_{\lambda,\nu} &= A_{\lambda+\nu} B_\lambda - A_\lambda B_{\lambda+\nu} \\ &= (A_\lambda A_{\nu-1,\lambda+1} + a_{\lambda+1} A_{\lambda-1} B_{\nu-1,\lambda+1}) B_\lambda \\ &\quad - A_\lambda (B_\lambda A_{\nu-1,\lambda+1} + a_{\lambda+1} B_{\lambda-1} B_{\nu-1,\lambda+1}) \\ &= a_{\lambda+1} B_{\nu-1,\lambda+1} (A_{\lambda-1} B_\lambda - A_\lambda B_{\lambda-1}) \\ &= (-1)^\lambda a_1 \cdots a_{\lambda+1} B_{\nu-1,\lambda+1}. \end{aligned}$$

□

Hence from (21) and (16),

$$|D_{\lambda,\nu}| = B_{\nu-1,\lambda+1} \leq B_{\lambda+\nu}, \tag{22}$$

which gives inequality (19).

References

1. F. Blumer, *Über die verschiedenen Kettenbruchentwicklungen beliebiger reeller Zahlen und die periodischen Kettenbruchentwicklungen quadratischer Irrationalitäten*, Acta Arith. 3, 1938, 3-63.
2. O. Perron, *Kettenbrüche*, Band 1, Teubner, 1954.
3. H. Tietze, *Über Kriterien für Konvergenz und Irrationalität unendlichen Kettenbrüche*, Math. Ann., 70, 1911, 236-265.