A CLASS OF GENERALIZED $3x + 1$ MAPPINGS OF BENOIT CLOITRE

KEITH MATTHEWS

1. Introduction

In an email to the author dated July 25, 2011, Benoit Cloitre described a mapping $F_m : \mathbb{Z} \to \mathbb{Z}$ which is similar to the well–known $3x + 1$ mapping, where $m \geq 2$ is an integer. First $f_m(x)$ is defined by

$$f_m(x) = \left\lfloor \left(\frac{m+1}{m} \right) x \right\rfloor. \tag{1}$$

Then

$$F_m(x) = \begin{cases} \frac{f_m(x)}{2} & \text{if } f_m(x) \text{ is even} \\ \frac{3f_m(x)+1}{2} & \text{if } f_m(x) \text{ is odd.} \end{cases} \tag{2}$$

If $m = 3, 5$ or $m \geq 7$, all trajectories $x, F_m(x), F_m^{(2)}(x) = F_m(F_m(x)), \ldots$, appear to eventually enter one of finitely many cycles; whereas if $m = 2, 4$ or 6, there appear to be divergent trajectories (i.e., with $|F_m^{(n)}(x)| \to \infty$), but again only finitely many cycles.

F_m can be regarded as a $2m$–branched generalized $3x + 1$ mapping ([1, p. 80]) for even m and an m–branched mapping for odd m. In both cases, we can use the Markov chain heuristics of [3] to predict everywhere cycling or the existence of divergent trajectories.

We remark that $0, 0$ and $-1, -1$ are always cycles, while $1, 2, 1$ is a cycle if $m \geq 3$ and $-4, -7, -4$ is a cycle if $m \geq 7$.

2. F_m as a d–branched mapping

If $x = mK + i, 0 \leq i < m$, then (1) gives

$$\begin{aligned} f_m(x) =& \left\lfloor \left(\frac{m+1}{m} \right) (mK + i) \right\rfloor \\ =& (m+1)K + i + \left\lfloor \left(\frac{m+1}{m} \right) i \right\rfloor \\ =& (m+1)K + i. \end{aligned} \tag{3}$$

Date: 12th August 2011.

If m is odd, then (3) gives $f_m(x) \equiv i \pmod 2$ and (2) implies

$$F_m(x) = \begin{cases} \frac{(m+1)K+i}{2} & \text{if } 0 \le i < m,\ i \text{ even} \\ \frac{(3(m+1)K+3i+1}{2} & \text{if } 0 \le i < m,\ i \text{ odd} \end{cases}$$

$$(4) \qquad = \begin{cases} (\frac{(m+1)}{2}x - \frac{i}{2})/m & \text{if } 0 \le i < m,\ i \text{ even} \\ (\frac{3(m+1)}{2}x - \frac{3i-m}{2})/m & \text{if } 0 \le i < m,\ i \text{ odd.} \end{cases}$$

Hence F_m is a d–branched mapping of the form

$$(5) \qquad F_m(x) = (m_i x - r_i)/d \text{ if } x \equiv i \pmod d, 0 \le i < d,$$

where $d = m$ and

$$(6) \qquad m_i = \begin{cases} (m+1)/2 & \text{if } 0 \le i < m,\ i \text{ even} \\ 3(m+1)/2 & \text{if } 0 \le i < m,\ i \text{ odd.} \end{cases}$$

If m is even, we write $x = 2mK + i, 0 \le i < 2m$ and (1) gives

$$(7) \qquad f_m(x) = \begin{cases} 2(m+1)K + i & \text{if } 0 \le i < m \\ 2(m+1)K + i + 1 & \text{if } m \le i < 2m. \end{cases}$$

Hence

$$f_m(x) \equiv \begin{cases} i \pmod 2 & \text{if } 0 \le i < m \\ i+1 \pmod 2 & \text{if } m \le i < 2m \end{cases}$$

and (2) implies

$$(8) \qquad F_m(x) = \begin{cases} ((m+1)x - i)/2m & \text{if } 0 \le i < m, i \text{ even} \\ (3(m+1)x + m - 3i)/2m & \text{if } 0 \le i < m, i \text{ odd} \\ (3(m+1)x + 4m - 3i)/2m & \text{if } m \le i < 2m, i \text{ even} \\ ((m+1)x + m - i)/2m & \text{if } m \le i < 2m, i \text{ odd} \end{cases}$$

and F_m is a d–branched mapping of the form (5), where $d = 2m$ and

$$(9) \qquad m_i = \begin{cases} m+1 & \text{if } 0 \le i < m,\ i \text{ even} \\ 3(m+1) & \text{if } 0 \le i < m,\ i \text{ odd} \\ 3(m+1) & \text{if } m \le i < 2m,\ i \text{ even} \\ m+1 & \text{if } m \le i < 2m,\ i \text{ odd.} \end{cases}$$

3. The case of m not divisible by 3

If 3 does not divide m, we see from (6) and (9) that $\gcd(m_i, d) = 1$ for $0 \le i < d$ and F_m is of relatively prime type ([1, p. 82]). Consequently we expect

(i) all trajectories will eventually enter a cycle if $m_0 \cdots m_{d-1} < d^d$;
(ii) almost all trajectories will be divergent if $m_0 \cdots m_{d-1} > d^d$;
(iii) the number of cycles will be finite.

Then if m is even, with $d = 2m$,

$$\begin{aligned} m_0 \cdots m_{d-1} < d^d &\iff (m+1)^m(3(m+1))^m < (2m)^{2m} \\ &\iff 3^m(m+1)^{2m} < (2m)^{2m} \\ &\iff 3(m+1)^2 < (2m)^2 \\ &\iff 0 < m^2 - 6m - 3 \\ &\iff 8 \le m, \end{aligned}$$

while if m is odd, with $d = m$,

$$\begin{aligned} m_0 \cdots m_{d-1} < d^d &\iff (\frac{m+1}{2})^{\frac{m+1}{2}}(\frac{3(m+1)}{2})^{\frac{m-1}{2}} < m^m \\ &\iff (1+1/m)^m 3^{\frac{m-1}{2}} < 2^m. \end{aligned}$$

Now $(1+1/m)^m < 3$ so

$$\begin{aligned} (1+1/m)^m 3^{\frac{m-1}{2}} &< 3^{\frac{m+1}{2}} \\ &< 2^m, \text{ if } m \ge 5. \end{aligned}$$

Hence if m is not divisible by 3, we expect

(i) every trajectory will eventually enter one of finitely many cycles if $m \ge 8$ is even or $m \ge 5$ is odd;
(ii) almost all trajectories will be divergent if $m = 2$ or 4.

Example 1. $m = 2$. We find experimentally, using the author's CALC program [2], only the cycles $0, 0$ and $-1, -1$. Also the trajectories starting with 1 and -7 appear to be divergent.

Example 2. $m = 8$. We find experimentally the following eight cycles :

(i) $0, 0$
(ii) $-1, -1$
(iii) $1, 2, 1$
(iv) $215, 362, \ldots, 383, 215$ (length 168)
(v) $680, 1148, \ldots, 1209, 680$ (length 21)
(vi) $595, 1004, \ldots, 1058, 595$ (length 21)
(vii) $663, 1118, \ldots, 1179, 663$ (length 21)
(viii) $49868, 84152, \ldots, 88655, 49868$ (length 21)

4. The case of m divisible by 3

Formulae (6) and (9) reveal that

$$\gcd(m_i, d) = \gcd(m_i, d^2) = \begin{cases} 1 & \text{if } m \text{ is odd and } m_i = (m+1)/2, \\ & \text{or } m \text{ is even and } m_i = m+1 \\ 3 & \text{if } m \text{ is odd and } m_i = 3(m+1)/2, \\ & \text{or } m \text{ is even and } m_i = 3(m+1). \end{cases}$$

Hence we are not dealing with a relatively prime mapping, but instead, one discussed in [3], where the behaviour of trajectories is determined by a Markov matrix

$Q(d) = [q_{ij}]$. We use a formula for q_{ij} which was only implicitly defined in Lemma 2.4 of [3, p. 32], namely

$$q_{ij} = \begin{cases} \gcd(m_i, d)/d & \text{if } F_m(i) \equiv j \pmod{\gcd(m_i, d)} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Now if m is odd,

$$F_m(i) = \begin{cases} i/2 & \text{if } 0 \le i < m,\ i \text{ even} \\ (3i+1)/2 & \text{if } 0 \le i < m,\ i \text{ odd,} \end{cases} \tag{11}$$

while if m is even,

$$F_m(i) = \begin{cases} i/2 & \text{if } 0 \le i < m,\ i \text{ even} \\ (3i+1)/2 & \text{if } 0 \le i < m,\ i \text{ odd} \\ (3i+4)/2 & \text{if } m \le i < 2m,\ i \text{ even} \\ i/2 & \text{if } m \le i < 2m,\ i \text{ odd.} \end{cases} \tag{12}$$

We find the following:

(i) If m is odd,

$$q_{ij} = \begin{cases} 1/m & \text{if } 0 \le i < m,\ i \text{ even} \\ 3/m & \text{if } 0 \le i < m,\ i \text{ odd},\ j \equiv 2 \pmod 3 \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

(ii) If m is even,

$$q_{ij} = \begin{cases} 1/m & \text{if } 0 \le i < m,\ i \text{ even} \\ 1/m & \text{if } m \le i < 2m,\ i \text{ odd} \\ 3/m & \text{if } 0 \le i < m,\ i \text{ odd},\ j \equiv 2 \pmod 3 \\ 3/m & \text{if } m \le i < 2m,\ i \text{ even},\ j \equiv 2 \pmod 3 \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Moreover $(Q(d))^2$ is a positive matrix.

Example 4. $m = 9$. Then (13) gives $Q(9) =$

$$\begin{bmatrix}
1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\
0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 0 & 1/3 \\
1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\
0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 0 & 1/3 \\
1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\
0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 0 & 1/3 \\
1/9 & /1/9 & 1/9 & /1/9 & 1/9 & /1/9 & 1/9 & /1/9 & 1/9 \\
0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 0 & 1/3 \\
1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9
\end{bmatrix}$$

Example 5. $m = 6$. Then (14) gives $Q(12) =$

$$\begin{bmatrix}
1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 \\
0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\
1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 \\
0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\
1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 \\
0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\
0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\
1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 \\
0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\
1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 \\
0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\
1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12
\end{bmatrix}$$

Experimentally, we found that when m is a multiple of 3, the stationary vector of $Q(d)$ is proportional to $X = (x_0, x_1, x_2, \ldots, x_{d-3}, x_{d-2}, x_{d-1})^t$, where for $0 \le t < d/3$,

$$(x_{3t}, x_{3t+1}, x_{3t+2}) = \begin{cases} (1,1,4) & \text{if } m = 6n \\ (2n+1, 2n+1, 8n+2) & \text{if } m = 12n+3 \\ (n+1, n+1, 4n+3) & \text{if } m = 12n+9. \end{cases}$$

Then for example, if $m = 12n + 9$,

$$\begin{aligned} m_0^{x_0} \cdots m_{d-1}^{x_{d-1}} &= \overbrace{RSRS \cdots RS}^{2n+1} R \\ &= R^{2n+2} S^{2n+1}, \end{aligned}$$

where

$$\begin{aligned} R &= \left(\frac{m+1}{2}\right)^{n+1} \left(\frac{3(m+1)}{2}\right)^{n+1} \left(\frac{m+1}{2}\right)^{4n+3} \\ S &= \left(\frac{3(m+1)}{2}\right)^{n+1} \left(\frac{m+1}{2}\right)^{n+1} \left(\frac{3(m+1)}{2}\right)^{4n+3}. \end{aligned}$$

Then

$$\begin{aligned} & \left(\frac{m_0}{d}\right)^{x_0} \cdots \left(\frac{m_{d-1}}{d}\right)^{x_{d-1}} < 1 \\ \iff & m_0^{x_0} \cdots m_{d-1}^{x_{d-1}} < d^{x_0 + \cdots + x_{d-1}} \\ \iff & R^{2n+2} S^{2n+1} < m^{(4n+3)(6n+5)} \\ \iff & \left(\frac{m+1}{2}\right)^{(6n+5)(4n+3)} 3^{3n+2)(4n+3)} < m^{(6n+5)(4n+3)} \\ \iff & ((1+1/m))^{(6n+5)}\, 3^{3n+2} < 2^{(6n+5)} \\ \iff & ((1+1/m))^{\frac{m+1}{2}}\, 3^{\frac{m-1}{2}} < 2^{\frac{m+1}{2}}. \end{aligned} \tag{15}$$

Now $(1 + 1/m)^m < 3$ and $1 + 1/m \le 3/2$ if $m \ge 2$. Hence

$$\begin{aligned} (1+1/m)^{m+1} 3^{\frac{m-1}{2}} &< 3(1+1/m) 3^{\frac{m-1}{2}} \\ &< 3^{\frac{m+1}{2}} 3/2. \end{aligned}$$

Hence (15) holds if

$$3^{\frac{m+1}{2}} 3/2 < 2^{m+1}$$

or equivalently

$$3^{m+3} < 4^{m+2}$$

and this holds for $m \ge 2$.

Hence we expect all trajectories to eventually cycle. Also it seems certain that there are only finitely many cycles for each such m. We have found seven cycles when $m = 9$:

(i) $0, 0$
(ii) $-1, -1$
(iii) $1, 2, 1$
(iv) $-4, -7, -4$
(v) $-6, -10, -6$

(vi) $-11, -19, -11$

(vii) $14, 23, 38, 21, 35, 19, 32, 53, 29, 16, 26, 14.$

References

[1] J.C. Lagarias, Ed. *The Ultimate Challenge: The* $3x+1$ *Problem*, AMS 2011.

[2] K. R. Matthews, `http://www.numbertheory.org/calc/krm_calc.html`, CALC, a number theory calculator.

[3] K. R. Matthews and A. M. Watts, *A Markov approach to the generalized Syracuse algorithm*, ibid. 45 (1985), 29–42.