ON THE CONTINUED FRACTIONS OF CONJUGATE QUADRATIC IRRATIONALITIES

KEITH R. MATTHEWS

We are dealing with the polynomial $f(x,y) = ax^2 + bxy + cy^2$, where $a > 0$ and $d = b^2 - 4ac > 0$ is not a square. Let $\rho = (-b + \sqrt{d})/2a$ and $\sigma = (-b - \sqrt{d})/2a$ be the roots of $f(x,1)$. The following was mentioned bt M. Pavone in 1986.

Proposition 1. (Lemma 5 of [3].) *Let* $\rho = [a_0, \ldots, a_m, \overline{b_1, \ldots, b_n}]$ *and* $\sigma = [c_0, \ldots, c_r, \overline{d_1, \ldots, d_n}]$ *be the roots of* f. *Also let* p_h/q_h *and* P_h/Q_h *denote the convergents of* ρ *and* σ*, respectively. We do not require the periods to have minimal lengths, but assume* m *and* r *are minimal, i.e.,* $a_m \neq b_n$ *and* $c_r \neq d_n$. *It is also convenient to assume* $n \geq 4$. *We let* $m = -1$ *if there is no preperiod.*

There exists an $i, 1 \leq i \leq 3$*, such that*

$$\sigma = [c_0, \ldots, c_r, \overline{b_{n-i}, \ldots, b_1, b_n, b_{n-1}, \ldots, b_{n-i+1}}]. \tag{1}$$

Also $i = 3$ *implies* $b_{n-1} = 1$. *When* $b_n = b_{n-1} = 1$*, then* $i = 3$ *if and only if* $m \geq 0$ *and* $r \geq 0$.

Proofs were not given by Pavone. We list all cases for (1) and give proofs. We need two lemmas.

Lemma 2. *If* $\xi = [a_0, a_1, \ldots]$*, then*

$$-\xi = \begin{cases} [-a_0 - 1, 1, a_1 - 1, a_2, \ldots] & \text{if } a_1 > 1; \\ [-a_0 - 1, a_2 + 1, a_3, \ldots] & \text{if } a_1 = 1. \end{cases} \tag{2}$$

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Remark 3. This is Lemma 3.1 of [1].

Lemma 4. *Let* $\rho = \frac{-b+\sqrt{d}}{2a} = [a_0, \ldots, a_m, \overline{b_1, \ldots, b_n}]$ *and* $\overline{\rho} = \frac{-b-\sqrt{d}}{2a}$. *Then*

$$\overline{\rho} = \begin{cases} [a_0, \ldots, a_m, -1, 1, b_n - 1, \overline{b_{n-1}, b_{n-2}, \ldots, b_1, b_n}] & \text{if } b_n > 1; \\ [a_0, \ldots, a_m, -1, b_{n-1} + 1, \overline{b_{n-2}, b_{n-3}, \ldots, b_1, b_n, b_{n-1}}] & \text{if } b_n = 1. \end{cases}$$

The preperiod $a_0, \ldots, a_m$ *can be absent.*

Proof. We have $\rho = [a_0, \ldots, a_m, \theta]$, where $\theta = [\overline{b_1, \ldots, b_n}]$. Taking conjugates gives

$$\begin{aligned} \overline{\rho} &= [a_0, \ldots, a_m, \overline{\theta}] \\ &= [a_0, \ldots, a_m, -[0, \overline{b_n, \ldots, b_1}]]. \end{aligned}$$

The desired conclusion now follows from Lemma 2. □

Our problem is to get rid of a negative partial quotient in the equations of Lemma 4. We use a matrix approach.

1. We first assume $b_n > 1$ and consider the identity

$$\begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_n - 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b_n - a_m & 1 \\ -1 & 0 \end{pmatrix}.$$

Case (a). Assume $m \geq 1$ and $b_n - a_m \geq 1$. We use the matrix product

$$\begin{pmatrix} a_{m-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_n - a_m & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a_{m-1}(b_n - a_m) - 1 & a_{m-1} \\ b_n - a_m & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_{m-1} - 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_n - a_m - 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence if $b_n > 1, m \geq 2$ and $b_n - a_m \geq 1$, we have

$$\sigma = \begin{cases} [a_0, \ldots, a_{m-2}, a_{m-1} - 1, 1, b_n - a_m - 1, \overline{b_{n-1} \ldots, b_1, b_n}] & \text{if } b_n - a_m > 1; \\ [a_0, \ldots, a_{m-2}, a_{m-1} - 1, 1 + b_{n-1}, \overline{b_{n-2} \ldots, b_1, b_n, b_{n-1}}] & \text{if } b_n - a_m = 1. \end{cases}$$

while if $b_n > 1, m = 1$ and $b_n - a_m \geq 1$, we have

$$\sigma = \begin{cases} [a_{m-1} - 1, 1, b_n - a_m - 1, \overline{b_{n-1} \ldots, b_1, b_n}] & \text{if } b_n - a_m > 1; \\ [a_{m-1} - 1, 1 + b_{n-1}, \overline{b_{n-2} \ldots, b_1, b_n, b_{n-1}}] & \text{if } b_n - a_m = 1. \end{cases}$$

On removing a zero partial quotient if $a_{m-1} = 1$, we obtain a continued fraction expansion for σ of the form (1), where $i = 1$ if $b_n - a_m > 1$, and $i = 2$ if $b_n - a_m = 1$.

Case (b). Assume $m \geq 1$ and $b_n - a_m = -b < 0$, or $m = 0$. Let $c = b_{n-1}$. Then

$$\begin{pmatrix} b_n - a_m & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b_{n-1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -b & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -bc + 1 & -b \\ -c & -1 \end{pmatrix} = -\begin{pmatrix} b - 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c - 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence if $b_n > 1, m \geq 1$ and $b_n - a_m < 0$, we have

$$\sigma = \begin{cases} [a_0, \ldots, a_{m-1}, a_m - b_n - 1, 1, b_{n-1} - 1, \overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}] & \text{if } b_{n-1} > 1; \\ [a_0, \ldots, a_{m-1}, a_m - b_n - 1, b_{n-2} + 1, \overline{b_{n-3}, \ldots, b_1, b_n, b_{n-1}, b_{n-2}}] & \text{if } b_{n-1} = 1. \end{cases}$$

while if $b_n > 1$ and $m = 0$, we have

$$\sigma = \begin{cases} [a_0 - b_n - 1, 1, b_{n-1} - 1, \overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}] & \text{if } b_{n-1} > 1; \\ [a_0 - b_n - 1, b_{n-2} + 1, \overline{b_{n-3}, \ldots, b_1, b_n, b_{n-1}, b_{n-2}}] & \text{if } b_{n-1} = 1. \end{cases}$$

On removing a zero partial quotient if $a_m - b_n = 1$ and $m \geq 1$, we obtain a continued fraction expansion for σ which has the form (1), where $i = 2$ if $b_{n-1} > 1$ and $i = 3$ if $b_{n-1} = 1$.

2. We now assume $b_n = 1$. Then

$$\begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_{n-1} + 1 & 1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} b_{n-1}(a_m - 1) - 1 & a_m - 1 \\ b_{n-1} & 1 \end{pmatrix}$$

$$= -\begin{pmatrix} a_m - 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_{n-1} - 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that $a_m \geq 2$ if $m \geq 1$. Hence if $b_n = 1$ and $m > 1$, we get the continued fraction

$$\sigma = \begin{cases} [a_0, \ldots, a_{m-1}, a_m - 2, 1, b_{n-1} - 1, \overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}] & \text{if } b_{n-1} > 1; \\ [a_0, \ldots, a_{m-1}, a_m - 2, 1 + +b_{n-2}, \overline{b_{n-3}, \ldots, b_1, b_n, b_{n-1}, b_{n-2}}] & \text{if } b_{n-1} = 1. \end{cases}$$

while if $b_n = 1$ and $m = 0$, we get the continued fraction

$$\sigma = \begin{cases} [a_0 - 2, 1, b_{n-1} - 1, \overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}] & \text{if } b_{n-1} > 1; \\ [a_0 - 2, 1 + +b_{n-2}, \overline{b_{n-3}, \ldots, b_1, b_n, b_{n-1}, b_{n-2}}] & \text{if } b_{n-1} = 1. \end{cases}$$

On removing a zero partial quotient if $a_m = 2$, we obtain the continued fraction expansion for σ in the form (1), with $i = 2$ if $b_{n-1} > 1$, and $i = 3$ if $b_{n-1} = 1$.

3. Finally, we consider the case $\rho = [\overline{b_1, \ldots, b_n}]$. Then by Lemma 4

$$\sigma = \begin{cases} [-1, 1, b_n - 1, \overline{b_{n-1}, \ldots, b_1, b_n}] & \text{if } b_n > 1; \\ [-1, b_{n-1} + 1, \overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}] & \text{if } b_n = 1. \end{cases}$$

Table 1 gives an expanded summary of all cases. There is similar table at [2] which assisted in the fine–tuning of Table 1.

References

[1] F. Halter–Koch, Principal ideals in quadratic orders, continued fractions and Diophantine equations, *J. Comb. Number Theory* **6** (2014) 17–30.

[2] F. Herzog, On the continued fractions of conjugate quadratic irrationalities, *Canad. Math. Bull.* **23** (1980) 199–206.

[3] M. Pavone, A Remark on a Theorem of Serret, *J. Number Theory* **23** (1986) 268–278.

	r	Cases	Continued fraction expansion of σ
$A11$	$m+1$	$b_n > 1, m \geq 2, b_n - a_m > 1, a_{m-1} > 1$	$[a_0, \ldots, a_{m-2}, a_{m-1} - 1, 1, b_n - a_m - 1, \overline{b_{n-1} \ldots, b_1, b_n}]$
$A12$	$m-1$	$b_n > 1, m > 2, b_n - a_m > 1, a_{m-1} = 1$	$[a_0, \ldots, a_{m-3}, a_{m-2} + 1, b_n - a_m - 1, \overline{b_{n-1} \ldots, b_1, b_n}]$
$A13$	1	$b_n > 1, m = 2, b_n - a_2 > 1, a_1 = 1$	$[a_0 + 1, b_n - a_2 - 1, \overline{b_{n-1} \ldots, b_1, b_n}]$
$A21$	m	$b_n > 1, m \geq 2, b_n - a_m = 1, a_{m-1} > 1$	$[a_0, \ldots, a_{m-2}, a_{m-1} - 1, 1 + b_{n-1}, \overline{b_{n-2} \ldots, b_1, b_n, b_{n-1}}]$
$A22$	$m-2$	$b_n > 1, m > 2, b_n - a_m = 1, a_{m-1} = 1$	$[a_0, \ldots, a_{m-3}, a_{m-2} + 1 + b_{n-1}, \overline{b_{n-2} \ldots, b_1, b_n, b_{n-1}}]$
$A23$	0	$b_n > 1, m = 2, b_n - a_2 = 1, a_1 = 1, a_0 \neq -1$	$[a_0 + 1 + b_{n-1}, \overline{b_{n-2} \ldots, b_1, b_n, b_{n-1}}]$
$A24$	-1	$b_n > 1, m = 2, b_n - a_2 = 1, a_1 = 1, a_0 = -1$	$[\overline{b_{n-1} \ldots, b_1, b_n}]$
$A25$	$m+2$	$b_n > 1, m \geq 1, a_m - b_n > 1, b_{n-1} > 1$	$[a_0, \ldots, a_{m-1}, a_m - b_n - 1, 1, b_{n-1} - 1, \overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}]$
$A251$	m	$b_n > 1, m \geq 2, a_m - b_n = 1, b_{n-1} > 1$	$[a_0, \ldots, a_{m-2}, a_{m-1} + 1, b_{n-1} - 1, \overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}]$
$A252$	1	$b_n > 1, m = 1, a_m - b_n = 1, b_{n-1} > 1$	$[a_0 + 1, b_{n-1} - 1, \overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}]$
$A26$	$m+1$	$b_n > 1, m \geq 1, a_m - b_n > 1, b_{n-1} = 1$	$[a_0, \ldots, a_{m-1}, a_m - b_n - 1, b_{n-2} + 1, \overline{b_{n-3}, \ldots, b_1, b_n, b_{n-1}, b_{n-2}}]$
$A261$	$m-1$	$b_n > 1, m \geq 2, a_m - b_n = 1, b_{n-1} = 1$	$[a_0, \ldots, {}_{m-2}, a_{m-1} + b_{n-2} + 1, \overline{b_{n-3}, \ldots, b_1, b_n, b_{n-1}, b_{n-2}}]$
$A262$	0	$b_n > 1, m = 1, a_m - b_n = 1, b_{n-1} = 1$	$[a_0 + b_{n-2} + 1, \overline{b_{n-3}, \ldots, b_1, b_n, b_{n-1}, b_{n-2}}]$
$B1$	2	$b_n > 1, m = 1, b_n - a_1 > 1$	$[a_0 - 1, 1, b_n - a_1 - 1, \overline{b_{n-1} \ldots, b_1, b_n}]$
$B2$	1	$b_n > 1, m = 1, b_n - a_1 = 1$	$[a_0 - 1, 1 + b_{n-1}, \overline{b_{n-2} \ldots, b_1, b_n, b_{n-1}}]$
$B21$	2	$b_n > 1, m = 0, b_{n-1} > 1$	$[-b_n + a_0 - 1, 1, b_{n-1} - 1, \overline{b_{n-2} \ldots, b_1, b_n, b_{n-1}}]$
$B22$	1	$b_n > 1, m = 0, b_{n-1} = 1$	$[-b_n + a_0 - 1, b_{n-2} + 1, \overline{b_{n-3} \ldots, b_1, b_n, b_{n-2}}]$
$C11$	$m+2$	$b_n = 1, b_{n-1} > 1, m \geq 1, a_m > 2$	$[a_0, \ldots, a_{m-1}, a_m - 2, 1, b_{n-1} - 1, \overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}]$
$C12$	$m+1$	$b_n = 1, b_{n-1} = 1, m \geq 1, a_m > 2$	$[a_0, \ldots, a_{m-1}, a_m - 2, b_{n-2} + 1, \overline{b_{n-3}, \ldots, b_1, b_n, b_{n-1}, b_{n-2}}]$
$C21$	m	$b_n = 1, b_{n-1} > 1, m > 1, a_m = 2$	$[a_0, \ldots, a_{m-2}, a_{m-1} + 1, b_{n-1} - 1, \overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}]$
$C22$	$m-1$	$b_n = 1, b_{n-1} = 1, m > 1, a_m = 2$	$[a_0, \ldots, a_{m-2}, a_{m-1} + b_{n-2} + 1, \overline{b_{n-3}, \ldots, b_1, b_n, b_{n-1}, b_{n-2}}]$
$C31$	1	$b_n = 1, b_{n-1} > 1, m = 1, a_1 = 2$	$[a_0 + 1, b_{n-1} - 1, \overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}]$
$C32$	0	$b_n = 1, b_{n-1} = 1, m = 1, a_1 = 2, a_0 \neq -1$	$[a_0 + b_{n-2} + 1, \overline{b_{n-3}, \ldots, b_1, b_n, b_{n-1}, b_{n-2}}]$
$C33$	-1	$b_n = 1, b_{n-1} = 1, m = 1, a_1 = 2, a_0 = -1$	$[\overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}]$
$D1$	2	$b_n = 1, b_{n-1} > 1, m = 0$	$[a_0 - b_n - 1, 1, b_{n-1} - 1, \overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}]$
$D2$	1	$b_n = 1, b_{n-1} = 1, m = 0$	$[a_0 - b_n - 1, b_{n-2} + 1, \overline{b_{n-3}, \ldots, b_1, b_n, b_{n-1}, b_{n-2}}]$
$G1$	2	$b_n > 1, m = -1$	$[-1, 1, b_n - 1, \overline{b_{n-1}, \ldots, b_1, b_n}]$
$G2$	1	$b_n = 1, m = -1$	$[-1, b_{n-1} + 1, \overline{b_{n-2}, \ldots, b_1, b_n, b_{n-1}}]$.

FIGURE 1. Table 1