The NSCF expansion of $\frac{p+q+\sqrt{p^2+q^2}}{p}$, $p > 2q > 0$.

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Theorem XIII of A.A.K. Ayyangar ([2, p.103]) states that the nearest square continued fraction expansion (NSCF) of $\xi_0 = \frac{p+q+\sqrt{p^2+q^2}}{p}, p > 2q > 0$ is purely recurring. In [1], it is stated that a period contains at most two complete quotients ξ_h of this form. This is proved in [2, p.109]. We give another proof. In the case where there are two complete quotients of the above form in a period, we show that they are separated by half a period-length. Our proof is an extension of the argument in [2, pp.112-113], which dealt with the case where there is only one complete quotient of the above form in a period.

Theorem. Let k be the period-length of the NSCF expansion of $\frac{p+q+\sqrt{p^2+q^2}}{p}$, where $p > 2q > 0$. Then if $\xi_h, 1 \leq h < k$ is also of this form,

(a) k is even and $h = k/2$;

(b) $Q_v = Q_{k-v}$ for $0 \leq v \leq k-1$; ($Q_0 = Q_{2h}$ by periodicity and $Q_1 = Q_{2h-1} = 2q$);

(c) $P_{k+1-v} = P_v$ for $2 \leq v \leq \frac{k}{2} - 1$;

(d) $b_v = b_{k-v}, 2 \leq v \leq \frac{k}{2} - 2, b_{k-1} - 1 = b_1, b_{\frac{k}{2}+1} = b_{\frac{k}{2}-1} - 1$, $b_{\frac{k}{2}} = 2$;

(e) $a_{k+1-v} = a_v$, $2 \leq v \leq \frac{k}{2} - 1$, $a_{\frac{k}{2}} = -1$, $a_{\frac{k}{2}+1} = 1$, $a_1 = 1, a_k = -1$.

By Corollary 4, [2, p.29], the period of the NSCF expansion of a quadratic surd has all partial denominators $b_\nu \geq 2$. Also in [2, pp.147-153], Perron introduces the transformation $\mathfrak{t}_1$, where if $a_v = -1$, $\frac{a_{v-1}|}{|b_{v-1}} - \frac{1|}{|b_v}$ is replaced by $\frac{a_{v-1}|}{|b_{v-1}-1} + \frac{1|}{|1} + \frac{1|}{|b_v-1}$. Repeated use of $\mathfrak{t}_1$ results in a transformation $\mathfrak{T}_1$.

Thus each partial numerator -1 produces a partial numerator 1, while each partial denominator b_v is replaced by:

$$\begin{array}{c} b_v \text{ if } a_v = 1, a_{v+1} = 1 \\ b_v - 1 \text{ if } a_v = 1, a_{v+1} = -1 \text{ or } a_v = -1, a_{v+1} = 1 \\ b_v - 2 \text{ if } a_v = -1, a_{v+1} = -1. \end{array}$$

Hence $\mathfrak{T}_1$ converts the NSCF expansion to a continued fraction where all partial numerators a_ν are 1 and all partial denominators b_ν are non-negative.

The following lemma shows that for a NSCF expansion where ξ_0 or ξ_1 are reduced surds, $\mathfrak{T}_1$ in fact produces the regular continued fraction (RCF) expansion of ξ_0.

Lemma 1. Suppose ξ_t and ξ_{t+1} are reduced quadratic surds. Then if $a_t = -1$ and $a_{t+1} = -1$, we have $b_t \geq 3$.

Proof. Assume ξ_t and ξ_{t+1} are reduced. Then

$$\begin{aligned} P_{t+1} &\geq Q_t + \tfrac{1}{2}Q_{t+1} && (1) \\ P_t &\geq Q_t + \tfrac{1}{2}Q_{t-1}. && (2) \end{aligned}$$

Then (1) and (2) give

$$b_tQ_t = P_{t+1} + P_t \geq 2Q_t + \tfrac{1}{2}Q_{t+1} + \tfrac{1}{2}Q_{t-1}.$$

Hence $b_tQ_t > 2Q_t$, as $Q_{t+1} > 0$ and $Q_{t-1} > 0$. Hence $b_t > 2$. Now suppose $\xi_h = \frac{P+Q+\sqrt{P^2+Q^2}}{P}, P > 2Q > 0$ occurs remotely in the cycle of ξ_0. We know that $a_h = -1$ and $a_{h+1} = 1$.

Also as $a_h = -1$, ξ_h gives rise to the RCF complete quotient $\xi_h - 1 = \frac{Q+\sqrt{P^2+Q^2}}{P}$ and by Lemma 2, section 5.5 of [2], this can only occur once in a period.

We now prove $h = k/2$.

Lemma 2. Suppose $\xi_0 = b_0 + \frac{a_1|}{|b_1} + \cdots + \frac{a_{k-1}|}{|b_{k-1}} + \cdots$ is purely periodic with period k and let

$$\zeta_v = -a_{k-v}/\overline{\xi}_{k-v} = \frac{P_{k-v} + \sqrt{D}}{Q_{k-v-1}},$$

for $v = 0, \ldots, k-1$ and where $a_0 = a_k$. Then if none of $\xi_{h+1}, \ldots, \xi_{k-1}$ has the form $\frac{p+q+\sqrt{p^2+q^2}}{p}, p > 2q > 0$, by Theorem VIII [2, p.98], we have the

Bhaskara expansion

$$\zeta_0 = b_{k-1} + \frac{a_{k-1}|}{|b_{k-2}} + \cdots + \frac{a_{h+2}|}{|\zeta_{k-h-2}}. \tag{3}$$

Now assume $\xi_0 = \frac{p+q+\sqrt{p^2+q^2}}{p}$, $p > 2q > 0$ and that $\xi_h = \frac{P+Q+\sqrt{P^2+Q^2}}{P}$, $P > 2Q > 0$, with $1 < h < k$. Then we know that none of $\xi_{h+1}, \ldots, \xi_{k-1}$ has this form and Lemma 2 applies. We find with $a_k = -1$ that

$$\zeta_0 = 1/\frac{p+q-\sqrt{p^2+q^2}}{p} = \frac{p+q+\sqrt{p^2+q^2}}{2q} = \frac{p-q+\sqrt{p^2+q^2}}{2q} + 1,$$

which together with (3), gives the Bhaskara expansion

$$\frac{p-q+\sqrt{p^2+q^2}}{2q} = b_{k-1} - 1 + \frac{a_{k-1}|}{|b_{k-2}} + \cdots + \frac{a_{h+2}|}{|\zeta_{k-h-2}}. \tag{4}$$

Now $\zeta_{k-h-2} = \frac{P_{h+2}+\sqrt{D}}{Q_{h+1}}$ and we see $P_{h+2} = (2a+1)Q - P$, where $a = \lfloor P/Q \rfloor$. Also $Q_{h+1} = 2Q$. Hence we have the Bhaskara expansion

$$\zeta_{k-h-2} = \frac{(2a+1)Q - P + \sqrt{D}}{2Q} = a + 1 - \frac{P+Q+\sqrt{D}}{P}. \tag{5}$$

Then equations (4) and (5) give

$$\frac{p-q+\sqrt{p^2+q^2}}{2q} = b_{k-1} - 1 + \frac{a_{k-1}|}{|b_{k-2}} + \cdots + \frac{a_{h+2}|}{|a+1} - \frac{1|}{|\frac{P+Q+\sqrt{D}}{P}}. \tag{6}$$

But

$$\frac{p+q+\sqrt{p^2+q^2}}{p} = 2 + \frac{2q}{p-q+\sqrt{D}},$$

so we also have the Bhaskara expansion

$$\frac{p-q+\sqrt{p^2+q^2}}{2q} = b_1 + \frac{a_2|}{|b_2} + \cdots + \frac{a_{k-h-1}|}{|b_{h-k-1}} + \frac{a_{k-h}|}{|\xi_{k-h}}. \tag{7}$$

By comparing (6) and (7), we deduce $\xi_{k-h} = \frac{P+Q+\sqrt{D}}{P}$ and so $k-h = h$, $k = 2h$. Also $b_{k-1} - 1 = b_1$, $a_h = -1$, $b_h = 2$, $b_{h-1} = a + 1 = b_{h+1} + 1$, the latter following from [3, pp.7-8].

Also $a_v = a_{k+1-v}$ for $v = 2, \ldots, h-1$ and $b_v = b_{k-v}$ for $v = 2, \ldots, h-2$. Finally, we have for $v = 2, \ldots, h-1$,

$$\xi_v = \zeta_{v-1},$$
$$\frac{P_v + \sqrt{D}}{Q_v} = \frac{P_{k-v+1} + \sqrt{D}}{Q_{k-v}},$$

so $P_v = P_{k+1-v}$ and $Q_v = Q_{k-v}$. The last equation also holds for $v = 0$ by periodicity and holds for $v = 1$ as $Q_1 = 2q = Q_k$.

Example. $\xi_0 = \frac{324+\sqrt{81770}}{283}$. Here $283^2 + 41^2 = 277^2 + 71^2 = 81770$. $p = 283, q = 41, P = 277, Q = 71, k = 8, k' = 10$.

$\xi_0 = \frac{324+\sqrt{81770}}{283}$	$a_1 = 1$	$b_0 = 2$
$\xi_1 = \frac{242+\sqrt{81770}}{82}$	$a_2 = 1$	$b_1 = 6$
$\xi_2 = \frac{250+\sqrt{81770}}{235}$	$a_3 = 1$	$b_2 = 2$
$\xi_3 = \frac{220+\sqrt{81770}}{142}$	$a_4 = -1$	$b_3 = 4$
$\xi_4 = \frac{348+\sqrt{81770}}{277}$	$a_5 = 1$	$b_4 = 2$
$\xi_5 = \frac{206+\sqrt{81770}}{142}$	$a_6 = 1$	$b_5 = 3$
$\xi_6 = \frac{220+\sqrt{81770}}{235}$	$a_7 = 1$	$b_6 = 2$
$\xi_7 = \frac{250+\sqrt{81770}}{82}$	$a_8 = -1$	$b_7 = 7$
$\xi_8 = \frac{324+\sqrt{81770}}{283}$	$a_9 = 1$	$b_8 = 2$

References

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[4] O. Perron, *Kettenbrüchen*, Band 1, Teubner, 1954.