THEORY OF THE NEAREST SQUARE CONTINUED FRACTION

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Abstract

This is a version of the (perhaps somewhat neglected) paper *Theory of The Nearest Square Continued Fraction*, A.A. Krishnaswami Ayyangar (AAK), J. Mysore Univ. **1**, (1941), 21-32, 97-117. The task was undertaken as the online version at http://www.ms.uky.edu/ sohum/AAK/PRELUDE.htm was poorly reproduced. Some of the explanations were hard to follow and have been expanded for the ease of the reader. Only Section 5.5.1 has not been vetted. The circle diagrams were kindly provided by Judy Matthews.

1. *Introduction*

The genesis of the present investigation is a remark of the late Sir Thomas Little Heath that the Indian Cyclic Method of solving the equation $x^2 - Ny^2 = 1$ in integers due to Bhaskara in 1150, is[1] 'remarkably enough, the *same* as that which was rediscovered and expounded by Lagrange in 1768'. We have pointed out elsewhere[2] that the Indian Cyclic Method implies a half-regular continued fraction (h.r.c.f. for brevity) with certain noteworthy properties which have not been previously investigated. If we remember that it was Lagrange who was mainly responsible for the neglect of the h.r.c.f. since he showed, by an example, how it would never uniformly lead to the

[1]See page 285, *Diophantus of Alexandria* by Sir T.L. Heath, Cambridge 1910

[2]See pages 602-604, *Curr. Sci.*, Vol. VI, No. 12, June 1938.

solution of the so-called Pellian equation, we can appreciate the distance between Lagrange's simple continued fraction and the one discussed in this paper.

This new continued fraction, we call, the *nearest square continued fraction* or *Bhaskara continued fraction* - (B.c.f. for brevity), the natural sequel to Bhaskara's cyclic method. The whole theory can be developed as it were from 'scratch' with the help of the simplest mathematics known to the Hindus about the fifth century A.D.

2. *The New Continued Fraction Defined*

2.1. Definition. A quadratic surd $\frac{P+\sqrt{R}}{Q}$ is said to be in *standard form* if R is a non-square positive integer and $P, Q, \frac{R-P^2}{Q}$ are integers, having no common factor other than 1.

Theorem I. If $a = \lfloor \frac{P+\sqrt{R}}{Q} \rfloor$, where $\frac{P+\sqrt{R}}{Q}$ is a standard surd and

$$\frac{P+\sqrt{R}}{Q} = a + \frac{Q'}{P'+\sqrt{R}} = a + 1 - \frac{Q''}{P''+\sqrt{R}},$$

then $\frac{P'+\sqrt{R}}{Q'}$ and $\frac{P''+\sqrt{R}}{Q''}$ are standard surds with the following properties:

(i) $P'' - P' = Q$; $P'' + P' = Q' + Q''$;
$Q' - \frac{1}{2}Q \leq P'$ if $Q' \leq Q''$;
$Q'' + \frac{1}{2}Q \leq P''$ if $Q'' \leq Q'$.

(ii) $Q'^2 + Q''^2 + Q^2 + 2Q'Q'' + 2QQ' - 2QQ'' = 4R$.

(iii) If $|Q'|, |Q''|, |Q|$ be all greater than $\sqrt{R}$, then
$|P'|, |P''|, \frac{1}{2}|Q|$ are all greater than $\sqrt{2R}$,
at least one of $|Q'|, |Q''|$ is less than $\frac{1}{2}|Q|$;
also $|P'|, |P''|, |Q'|, |Q''|$ are less than $|Q|$.

(iv) (a) If $|Q| < 2\sqrt{R}$, then $Q' > 0$ and $Q'' > 0$ and at least one of them is less than $\sqrt{R}$;
(b) if $|Q| < \sqrt{2R}$, then one of P', P'' is positive;
(c) if $|Q| < \sqrt{R}$, then $0 < P' < 2\sqrt{R}$ and $0 < P'' < 2\sqrt{R}$.

(v) (a) If $|Q| < 2\sqrt{R}$, then $Q' \gtreqless Q''$ according as

$$\frac{Q'}{P' + \sqrt{R}} \gtreqless \tfrac{1}{2} + \frac{\sqrt{R}}{Q} - \frac{\sqrt{4R - Q^2}}{2Q};$$

(b) if $|Q| > 2\sqrt{R}$, then $|Q'| \gtreqless |Q''|$ according as

$$\frac{Q'}{P' + \sqrt{R}} \gtreqless \tfrac{1}{2} + \frac{\sqrt{R}}{Q}.$$

Proof. (i) and (ii) follow readily from the relations:-

$$\begin{array}{llll} (1) & P' = aQ - P; & (2) & P'' = (a+1)Q - P, \\ (3) & P'^2 = R - QQ'; & (4) & {P''}^2 = R + QQ''. \end{array}$$

The elements of the triple $(\frac{R-P^2}{Q}, P, Q)$ can be expressed as the sum of integral multiples of the elements of the triple $(\frac{R-P'^2}{Q'}, P', Q')$ and *vice-versa*. Hence $\frac{P'+\sqrt{R}}{Q'}$ is also a standard surd, when $\frac{P+\sqrt{R}}{Q}$ is one. Similarly $\frac{P''+\sqrt{R}}{Q''}$ is also standard.

From (ii),

$$\begin{aligned} (Q' - Q'' + Q)^2 + 4Q'Q'' &= 4R = (Q' + Q'' + Q)^2 - 4QQ'' \\ &= (Q' + Q'' - Q)^2 + 4QQ'' \qquad (5) \end{aligned}$$

If $|Q|, |Q'|, |Q''|$ all be greater than $\sqrt{R}$, then $|QQ'|, |QQ''|, |Q'Q''| > R$ and this implies $Q'Q'' < 0, QQ'' > 0$ and $QQ' < 0$.

Hence Q, Q'' are of the same sign and different from that of Q'. (α).

Again, $\frac{Q'}{P'+\sqrt{R}}$ and $\frac{Q''}{P''+\sqrt{R}}$ are positive proper fractions, so that if Q', Q'' area of opposite signs, so also are the pairs $P' + \sqrt{R}$ and $P'' + \sqrt{R}$ and P', P''; and the latter are absolutely greater than $\sqrt{R}$.

From (3), (4) and (α), $|P'| > \sqrt{2R}, |P''| > \sqrt{2R}$. (β).

From (i) and (β), $|Q| = |P'| + |P''| > 2\sqrt{2R}$ and one of P', P'' is not absolutely greater than $\frac{1}{2}|Q|$.

From (3) and (α), $|QQ'| < P'^2$; but $|Q| > |P'|$. Hence $|Q'| < |P'| < |Q|$; similarly $|Q''| < |P''| < |Q|$.

Hence $|Q'|$ or $|Q''|$ is less than $\frac{1}{2}|Q|$, according as $|P'|$ or $|P''|$ is not greater than $\frac{1}{2}|Q|$. This proves (iii).

If $|P'|, |P''|$ be both less than $\sqrt{R}$, we have from (3) and (4), Q, Q' of the same sign and different from that of Q''.

By (β), $P' + \sqrt{R}$ and $P'' + \sqrt{R}$ must also be of oppositie signs, which contradicts the assumption that $|P'|, |P''| < \sqrt{R}$.

Hence $|P'|, |P''|$ are never both less than $\sqrt{R}$. $\qquad (\gamma)$

If $|P'|, |P''|$ are both greater than $\sqrt{R}$, then $|Q| > 2\sqrt{R}$.

If $|Q| < 2\sqrt{R}$, one of $|P'|, |P''|$ is less and the other greater than $\sqrt{R}$, so that by (3) and (4), Q', Q'' are of the same sign.

When Q', Q'' are of the same sign, $P' + \sqrt{R}$ and $P'' + \sqrt{R}$ are also of the same sign and the numerically greater of $|P'|, |P''|$ must be positive, and so all the quantiities $P' + \sqrt{R}, P'' + \sqrt{R}, Q', Q''$ must be positive.

Therefore $Q'Q'' < R$ by (5) and so one of Q', Q'' is less than $\sqrt{R}$.

Again, if either $Q' < Q'', P' < 0, P'' > 0$, or $Q' > Q'', P'' < 0, P' > 0$, we have $Q(Q' - Q'') = (P'' - P')(Q' - Q'') < 0$, and by (ii) and (i), $Q^2 + (Q' + Q'')^2 > 4R$ and $|Q| > |Q' + Q''|$ and therefore $Q^2 > 2R$.

If $Q' = Q''$ and P', P'' be of opposite signs, then $Q^2 + (Q' + Q'')^2 = 4R$ and again $Q^2 > 2R$.

Therefore, when $|Q| < \sqrt{2R}$, we must have P' or P'' or both positive, according as $Q' < Q''$ or $Q' > Q''$ or $Q' = Q''$.

From (3), $|QQ'| = |\sqrt{R} - P'| \cdot |\sqrt{R} + P'|$. But $|Q'| < |P' + \sqrt{R}|$, so $|Q| > |\sqrt{R} - P'|$.

If $\sqrt{R} > |Q|$, then $\sqrt{R} > |\sqrt{R} - P'|$, so $2\sqrt{R} > P' > 0$. Similarly $2\sqrt{R} > P'' > 0$. Thus (iv) is proved.

If $|Q| < 2\sqrt{R}$, we have from (ii), $(Q' + Q'')^2 \gtrless 4R - Q^2$ according as $Q(Q' - Q'') \lesseqgtr 0$. By (iv), $Q', Q'' > 0$ and if $Q < 0$ and $Q' > Q''$, we have $(Q' + Q'')^2 > 4R - Q^2$,

$$\text{i.e., } \frac{Q' + Q''}{-2Q} > \sqrt{\frac{R}{Q^2} - \tfrac{1}{4}}$$

$$\text{i.e., } \frac{2P' + Q}{-2Q} > \sqrt{\frac{R}{Q^2} - \tfrac{1}{4}}$$

$$\text{i.e., } \frac{\sqrt{R} - P'}{Q} > \tfrac{1}{2} + \frac{\sqrt{R}}{Q} + \sqrt{\frac{R}{Q^2} - \tfrac{1}{4}}$$

$$\text{i.e., } \frac{Q'}{P' + \sqrt{R}} > \tfrac{1}{2} + \frac{\sqrt{R}}{Q} - \frac{\sqrt{4R - Q^2}}{2Q}.$$

The same result is obtained when $Q > 0$ and $Q' > Q''$.

Hence, when $Q' > Q''$ and $|Q| < 2\sqrt{R}$,

$$\frac{Q'}{P' + \sqrt{R}} > \tfrac{1}{2} + \frac{\sqrt{R}}{Q} - \frac{\sqrt{4R - Q^2}}{2Q}.$$

Similarly , when $Q' \leq Q''$, we can prove that

$$\frac{Q'}{P' + \sqrt{R}} \leq \tfrac{1}{2} + \frac{\sqrt{R}}{Q} - \frac{\sqrt{4R - Q^2}}{2Q}.$$

Again, from (ii), if $|Q| > 2\sqrt{R}$, $Q(Q' - Q'') < 0$,

$$\text{i.e., } \frac{Q' + Q''}{Q} \gtrless 0 \text{ according as } |Q'| \lessgtr |Q''|,$$

$$\text{i.e., } \frac{2P' + Q}{2Q} \gtrless 0 \text{ according as } |Q'| \lessgtr |Q''|,$$

$$\text{i.e., } \tfrac{1}{2} + \frac{\sqrt{R}}{Q} \gtrless \frac{\sqrt{R} - P'}{Q} \text{ according as } |Q'| \lessgtr |Q''|,$$

$$\text{i.e., } \tfrac{1}{2} + \frac{\sqrt{R}}{Q} \gtrless \frac{Q'}{P' + \sqrt{R}} \text{ according as } |Q'| \lessgtr |Q''|.$$

If $|Q'| = |Q''|$, then $Q' \neq Q''$ and therefore $Q' + Q'' = 0$, which implies $\frac{Q'}{P'+\sqrt{R}} = \frac{1}{2} + \frac{\sqrt{R}}{Q}$. Thus (v) is proved.

2.2. Having settled the preliminaries, we proceed to define the new continued fraction development as follows:

Let $\xi_0 = \frac{P+\sqrt{R}}{Q}$ be a surd in standard form and $a = \lfloor \xi_0 \rfloor$. Then ξ_0 can be represented in one of two forms

$$\xi_0 = a + \frac{Q'}{P' + \sqrt{R}} \quad \text{(I) or } \xi_0 = a + 1 - \frac{Q''}{P'' + \sqrt{R}} \qquad \text{(II)},$$

where $\frac{P'+\sqrt{R}}{Q'}$ and $\frac{P''+\sqrt{R}}{Q''}$ are also standard surds.

We call (I) the *positive* and (II) the *negative* representation of ξ_0. Choose the partial denominator b_0 and numerator ϵ_1 of the new continued fraction development:

(a) $b_0 = a$ if $|Q'| < |Q''|$, or $|Q'| = |Q''|$ and $Q < 0$, with $\epsilon_1 = 1$

(b) $b_0 = a + 1$ if $|Q'| > |Q''|$, or $|Q'| = |Q''|$ and $Q > 0$, with $\epsilon_1 = -1$

Then $\xi_0 = \frac{P+\sqrt{R}}{Q} = b_0 + \frac{\epsilon_1}{\xi_1}$, where $|\epsilon_1| = 1, b_0$ an integer and $\xi_1 = \frac{P_1+\sqrt{R}}{Q_1} > 1$. Also $P_1 = P'$ or P'' and $Q_1 = Q'$ or Q'', according as $\epsilon_1 = 1$ or -1.

We proceed similarly with ξ_1 and so on. Then

$$\xi_n = b_n + \frac{\epsilon_{n+1}}{\xi_{n+1}} \text{ and } \xi_0 = b_0 + \frac{\epsilon_1|}{|b_1} + \frac{\epsilon_2|}{|b_2} + \cdots \tag{1}$$

This development is called the *Bhaskara continued fraction* (B.c.f), or *nearest square continued fraction* for reasons to be noted presently.

Analogous classical relations connecting integers $P_n, Q_n, P_{n+1}, Q_{n+1}$ are

$$P_{n+1} + P_n = b_n Q_n \tag{2}$$

$$P_{n+1}^2 + \epsilon_{n+1} Q_n Q_{n+1} = R. \tag{3}$$

By Theorem I (iii), the $|Q_n|$ successively diminish as long as $|Q_n| > \sqrt{R}$ and so ultimately, we have $|Q_n| < \sqrt{R}$. When this stage is reached, the P_n and Q_n thereafter become positive and bounded , $P_n < 2\sqrt{R}, Q_n < \sqrt{R}$ by Theorem I (iv). This implies eventual periodicity of the complete quotients and thence the partial quotients. Hence we have

Theorem II. Every B.c.f. development of a quadratic surd is an eventually periodic half-regular continued fraction (h.r.c.f).

Note. (1) If $\xi_0 = b_0 + \frac{\epsilon_1|}{|b_1} + \frac{\epsilon_2|}{|b_2} + \cdots$ is a B.c.f., then so is $-\xi_0 = -b_0 - \frac{\epsilon_1|}{|b_1} + \frac{\epsilon_2|}{|b_2} + \cdots$. This follows immediately from the manner of the development, which takes into account the relative magnitudes and not the signs of the Q_n.

(2) From Theorem I (iv), it is easily seen that

(i) $\epsilon_{n+1} = 1$, if $\{\xi_n\} < \frac{1}{2}$ and $Q_n > 0$;

(ii) $\epsilon_{n+1} = -1$, if $\{\xi_n\} > \frac{1}{2}$ and $Q_n < 0$.

(3) From Theorem I (iii), it follows that if $2^{n-1}\sqrt{R} < |Q| < 2^n\sqrt{R}$, then $0 < |Q_m| < \sqrt{R}$ for some value of m with $n < m < 1 + \log_2|Q| - \frac{1}{2}\log_2 R$.

2.3. *Implications in the Conditions of the Definition of the B.c.f.*

If $\frac{P+\sqrt{R}}{Q} = a + \frac{Q'}{P'+\sqrt{R}} = a + 1 - \frac{Q''}{P''+\sqrt{R}}$ as in §2.2, we have

$$|Q'| \lesseqgtr |Q''| \iff |QQ'| \lesseqgtr |QQ''| \iff |P'^2 - R| \lesseqgtr |P''^2 - R|. \qquad (4)$$

Hence, if we are choosing the lesser of $|Q'|$ and $|Q''|$, we are choosing, in effect, the nearer of the two squares P'^2 and P''^2 to R as the basis of our development; and if the two squares are equidistant from R, we can obviously choose either; but to avoid ambiguity, we observe the convention that we choose Q' or Q'', according as $Q < 0$ or $Q > 0$.

Thus, the name *nearest square continued fraction* is justified.

With the help of Theorem I (v), we may give the following alternative choice rule: we assign to each complete quotient $\frac{P_n+\sqrt{R}}{Q_n}$ a positive or negative representation, according as its fractional part is $<$ or $>$ than $\frac{1}{2}+\frac{\sqrt{R}}{Q_n}-\frac{\sqrt{4R-Q_n^2}}{2Q_n}$ (resp. $\frac{1}{2}+\frac{\sqrt{R}}{Q_n}$), with $|Q_n|$ being $<$ (resp. $>$) $2\sqrt{R}$.

When the fractional part is equal to $\frac{1}{2}+\frac{\sqrt{R}}{Q_n}-\frac{\sqrt{4R-Q_n^2}}{2Q_n}$ (or $\frac{1}{2}+\frac{\sqrt{R}}{Q_n}$), which we may call *critical fractions*, the representation is chosen positive or negative, according as Q_n is negative or positive.

Such a representation is called a *Bhaskara representation* (B.R.).

If $\frac{P+\sqrt{R}}{Q} = b_0 + \frac{\epsilon_1 Q_1}{P_1+\sqrt{R}}$ be a Bhaskara representation, where $0 < |Q| < \sqrt{R}$. Then by (4) above

$$|{P_1}^2 - R| \leq |(P_1 + \epsilon_1 Q)^2 - R|. \qquad (A)$$

Exercise (KRM) Noting that $|Q| < \sqrt{R}$ implies $Q_1 > 0$, prove that (A) is equivalent to $\frac{1}{2}Q_1 \leq |P_1 + \frac{1}{2}\epsilon_1 Q - \frac{1}{2}Q_1|$. Use Theorem I(i) to prove that $P_1 + \frac{1}{2}\epsilon_1 Q - \frac{1}{2}Q_1 > 0$ and deduce that (A) is equivalent to $Q_1 - \frac{1}{2}\epsilon_1 Q \leq P_1$.

From Theorem I (i), we get $Q_1 - \frac{1}{2}\epsilon_1 Q \leq P_1$; if the *l.h.s.* of this be negative, $|Q_1 - \frac{1}{2}\epsilon_1 Q| \leq P_1$, since $Q_1^2 + \frac{1}{4}Q^2 < \frac{1}{2}Q^2 < R$, Q_1 being less than $\frac{1}{2}|Q|$.

Thus, $Q_1 - \frac{1}{2}\epsilon_1 Q \leq P_1$ implies $|Q_1 - \frac{1}{2}\epsilon_1 Q| \leq P_1$; and *vice versa.* (B)

Squaring both sides of the above inequality, we get

$$Q_1^2 + \tfrac{1}{4}Q^2 \leq P_1^2 + \epsilon_1 Q Q_1 = R, \text{ i.e., } Q_1^2 + \tfrac{1}{4}Q^2 \leq R. \qquad \text{(C)}$$

Conversely, it is seasy to see that (C) implies (B).

Hence, (A),(B),(C) are all equivalent to one another, when $0 < |Q| < \sqrt{R}$.

‡ Similarly, we can write down another set of equivalent conditions :

$$|P_1^2 - R| \leq |(P_1 + \epsilon_1' Q_1)^2 - R|. \quad (A'); \quad |Q - \tfrac{1}{2}\epsilon_1 Q_1| \leq P_1. \quad (B');$$
$$Q^2 + \tfrac{1}{4}Q_1^2 \leq R. \quad (C').$$

It is not difficult to verify that (C) and (C') imply that P_1 and $|P_1 + \epsilon_1' Q_1|$ (or, $|P_1 + \epsilon_1 Q|$) are such that one is less and the other greater than $\sqrt{R}$. (D)

Further, if one of the equivalent pairs (A),(A'); (B),B'); (C),(C') implying (D) holds, the following inequalities are true :

$$P_1 \geq \tfrac{1}{2}|Q|, \tfrac{1}{2}Q_1. \quad \text{(E)}; \quad |P_1 - \sqrt{R}| < |Q|, Q_1. \quad \text{(F)}$$

For, (E) is evident when $\epsilon_1 Q = -|Q|$; and when $\epsilon_1 Q = |Q|$ and $|Q| \geq Q_1$, (B') shows $P_1 \geq |Q| - \frac{1}{2}Q_1 \geq \frac{1}{2}|Q| \geq \frac{1}{2}Q_1$; when $\epsilon_1 Q = |Q|$ and $|Q| \leq Q_1$, we get the same result from (B). (F) follows immediately from (D), for eample, if $P_1 < \sqrt{R}$, then $|P_1 + \epsilon_1 Q|$ and $|P_1 + \epsilon_1' Q_1|$ are both greater than $\sqrt{R}$ and ϵ_1' in this case must be $+1$.

That the condition (C') can co–exist with (C) is clear from the condera-tion that the Q's in the B.c.f. development ultimately become positive and satisfy the conditions (A),(B), or (C). Since the Q's cannot go on perpetually decreasing after they become positive, a stage must come when a Q is not less than its predecessor. Thus, if $Q_1 \geq Q$, we get

$$Q^2 + \tfrac{1}{4}Q_1^2 \leq Q_1^2 + \tfrac{1}{4}Q^2 \leq R.$$

3. *Characteristics of the Ultimate Partial and Complete Quotients*

‡ $\epsilon_1' = +1$ or -1 according as $P_1 < \sqrt{R}$ or $P_1 > \sqrt{R}$.

Definition. A surd in the standard form $\frac{P_v+\sqrt{R}}{Q_v}$ is said to be a *special* surd, when its successor $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}}$ in the B.c.F. development is such that

$$Q_{v+1}^2 + \tfrac{1}{4}Q_v^2 \leq R \text{ and } Q_v^2 + \tfrac{1}{4}Q_{v+1}^2 \leq R.$$

A surd is said to be *semi-reduced* if it is the successor of a special surd. The successor of a semi-reduced surd is called a *reduced* surd.

Theorem III. The conjugate of a semi-reduced surd has its absolute value less than 1.

Proof. The conjugate of the semi-reduced surd $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}}$ is $\frac{P_{v+1}-\sqrt{R}}{Q_{v+1}}$, whose absolute value is less than 1 by §2.3 (F).

Theorem IV. A semi-reduced surd is also a special surd.

Proof. Let $\frac{P_v+\sqrt{R}}{Q_v}$ be a special surd and $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}}$ its successor, with B.R.'s given by

$$\frac{P_v+\sqrt{R}}{Q_v} = b_v + \frac{\epsilon_{v+1}Q_{v+1}}{P_{v+1}+\sqrt{R}};$$
$$\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}} = b_{v+1} + \frac{\epsilon_{v+2}Q_{v+2}}{P_{v+2}+\sqrt{R}}.$$

It is required to prove that

$$\text{(i) } Q_{v+1}^2 + \tfrac{1}{4}Q_{v+2}^2 \leq R \text{ and (ii) } Q_{v+2}^2 + \tfrac{1}{4}Q_{v+1}^2 \leq R. \qquad (1)$$

Only (i) needs proof, as $0 < Q_{v+1} < \sqrt{R}$ and inequality (C) of the Lemma, imply (ii).

Now (i) is true when $Q_{v+1} \leq Q_{v+2}$ or $Q_{v+2} \leq |Q_v|$. The former follows as $Q_{v+1} \leq Q_{v+2}$ implies

$$Q_{v+1}^2 + \tfrac{1}{4}Q_{v+2}^2 \leq Q_{v+2}^2 + \tfrac{1}{4}Q_{v+1}^2 \leq R.$$

While if $Q_{v+2} \leq |Q_v|$, then

$$Q_{v+1}^2 + \tfrac{1}{4}Q_{v+2}^2 \leq Q_{v+1}^2 + \tfrac{1}{4}Q_v^2 \leq R,$$

We have therefore to consider only the remaining case

$$Q_{v+1} > Q_{v+2} > |Q_v|. \qquad (2)$$

Since $P_{v+2}^2 - P_{v+1}^2 = Q_{v+1}(\epsilon_{v+1}Q_v - \epsilon_{v+2}Q_{v+2})$ and $P_{v+2} + P_{v+1} = b_{v+1}Q_{v+1}$, we have

$$P_{v+2} - P_{v+1} = (\epsilon_{v+1}Q_v - \epsilon_{v+2}Q_{v+2})/b_{v+1}. \qquad (3)$$

If $b_{v+1} = 1$, we have

$$P_{v+1} = \tfrac{1}{2}Q_{v+1} - \tfrac{1}{2}\epsilon_{v+1}Q_v + \tfrac{1}{2}\epsilon_{v+2}Q_{v+2} < Q_{v+1} - \tfrac{1}{2}\epsilon_{v+1}Q_v.$$

But by hypothesis, $Q_{v+1}^2 + \frac{1}{4}Q_v^2 \leq R$, so

$$P_{v+1} \geq Q_{v+1} - \tfrac{1}{2}\epsilon_{v+1}Q_v. \qquad (4)$$

Thus there is a contradiction. Hence

$$b_{v+1} \geq 2. \qquad (4')$$

From (3) and (4′),

$$|P_{v+2} - P_{v+1}| \leq \tfrac{1}{2}|\epsilon_{v+1}Q_v - \epsilon_{v+2}Q_{v+2}|.$$

If $P_{v+2} \leq P_{v+1}$, then

$$\begin{aligned} P_{v+2} &\geq P_{v+1} + \tfrac{1}{2}\epsilon_{v+1}Q_v - \tfrac{1}{2}\epsilon_{v+2}Q_{v+2} \\ &\geq Q_{v+1} - \tfrac{1}{2}\epsilon_{v+2}Q_{v+2} \text{ by (4)}, \end{aligned}$$

which is what we have to prove, being equivalent to (1).

Next, if $P_{v+2} > P_{v+1}$, then

$$P_{v+2} - P_{v+1} \leq \tfrac{1}{2}(\epsilon_{v+1}Q_v - \epsilon_{v+2}Q_{v+2}), \qquad (5)$$

Since both sides of (5) are positive, $\epsilon_{v+2} = -1$, lest (2) should be contradicted.

Now, two cases may occur: either

$$P_{v+2} + \tfrac{1}{2}\epsilon_{v+2}Q_{v+2} \geq Q_{v+1} \text{ or } < Q_{v+1},$$

the latter of which will be proved to be impossible.

For in the latter case,

$$\begin{aligned} P_{v+2} &< \tfrac{1}{2}Q_{v+2} + Q_{v+1}, \text{ since } \epsilon_{v+2} = -1, \\ &< \tfrac{3}{2}Q_{v+1} \text{ by (2)}. \end{aligned}$$

Hence $P_{v+1} + P_{v+2} < 2P_{v+2} < 3Q_{v+1}$, so that $b_{v+1} = 1$ or 2.

But by (4′), $b_{v+1} \geq 2$, so $b_{v+1} = 2$.

Then from (3)

$$\begin{aligned} P_{v+1} &= Q_{v+1} - \tfrac{1}{4}\epsilon_{v+1}Q_v - \tfrac{1}{4}Q_{v+2} \\ &\geq Q_{v+1} - \tfrac{1}{2}\epsilon_{v+1}Q_v \text{ by (4).} \end{aligned}$$

Hence $Q_{v+2} \leq \epsilon_{v+1}Q_v$, which will be impossible if the right-hand side is negative and will contradict (2) if the right-hand side is positive.

Hence $P_{v+2} + \frac{1}{2}\epsilon_{v+2}Q_{v+2} \geq Q_{v+1}$ and our theorem is established.

Corollary 1. The successor of a reduced surd is a reduced surd.

Corollary 2. All the complete quotients of a B.c.f. are ultimately reduced surds.

Corollary 3. The conjugate of a reduced surd has its absolute value less than 1 .

Corollary 4. The partial denominators corresponding to a semi-reduced and therefore a reduced surd, are always greater than 1.

Proof. For by (E) §2.3, we have $P_{v+2} > \frac{1}{2}Q_{v+1}$ and $P_{v+1} > \frac{1}{2}Q_{v+1}$, so that $P_{v+2} + P_{v+1} > Q_{v+1}$, i.e., $b_{v+1}Q_{v+1} > Q_{v+1}$. Hence $b_{v+1} > 1$.

Theorem V. A semi-reduced surd $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}}$ is greater than $\frac{1+\sqrt{5}}{2}$.

Proof. If $Q_{v+1} \leq \frac{\sqrt{4R}}{\sqrt{5}}$, then $\frac{\sqrt{R}}{Q_{v+1}} \geq \frac{\sqrt{5}}{2}$.

But $P_{v+1} \geq \frac{1}{2}Q_{v+1}$, so

$$\frac{P_{v+1} + \sqrt{R}}{Q_{v+1}} \geq \frac{1 + \sqrt{5}}{2}.$$

If $P_{v+1} \geq Q_{v+1}$, obviously $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}} > 2 > \frac{1+\sqrt{5}}{2}$.

If $P_{v+1} < Q_{v+1}$ and $Q_{v+1} > \frac{\sqrt{4R}}{\sqrt{5}}$, then

$$2.118\cdots = 1 + \frac{\sqrt{5}}{2} > \frac{P_{v+1} + \sqrt{R}}{Q_{v+1}} > 1.$$

But $b_{v+1} \geq 2$. Hence either

$$\frac{P_{v+1} + \sqrt{R}}{Q_{v+1}} > 2 \text{ or else } 1 < \frac{P_{v+1} + \sqrt{R}}{Q_{v+1}} < 2.$$

In the latter case, its Bhaskara representation must be negative, so that its fractional part, by Theorem I (iii) is greater than or equal to the critical fraction

$$\tfrac{1}{2} + \frac{\sqrt{R}}{Q_{v+1}} - \frac{\sqrt{4R - Q_{v+1}^2}}{2Q_{v+1}},$$

which is greater than $\frac{\sqrt{5}-1}{2}$ when $\frac{\sqrt{R}}{Q_{v+1}} < \frac{\sqrt{5}}{2}$.[3] In this case

$$\begin{aligned}
\frac{P_{v+1} + \sqrt{R}}{Q_{v+1}} &= 2 - \frac{Q_{v+2}}{P_{v+2} + \sqrt{R}} \\
&= 1 + (\text{a fraction greater than } \frac{\sqrt{5}-1}{2}) \\
&> \frac{\sqrt{5}+1}{2}.
\end{aligned}$$

Thus in all cases, $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}} \geq \frac{\sqrt{5}+1}{2}$.

Moreover strict inequality occurs, as $\frac{\sqrt{5}+1}{2}$ is not semi-reduced.

Corollary 1. A reduced surd is always greater than $\frac{1+\sqrt{5}}{2}$.

Corollary 2. All the complete quotients of a B.c.f. are ultimately and therefore in the recurring cycle, greater than $\frac{1+\sqrt{5}}{2}$.

Hence we have

Theorem VI. The cyclic part of the Bhaskara continued fraction is *canonical.* ‖

Theorem VII. If $\frac{P_v+\sqrt{R}}{Q_v}$ and $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}}$ are successive reduced surds, then $|P_{v+1}-P_v| \leq Q_v$. Moreover equality occurs if and only if $\frac{P_v+\sqrt{R}}{Q_v} = \frac{b_v-1+\sqrt{b_v^2+1}}{2}$ and $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}} = \frac{b_v+1+\sqrt{b_v^2+1}}{b_v}$, where $b_v > 2$.

---

[3] Let $\theta = \sqrt{R}/Q_{v+1}$. Then

$$\begin{aligned}
&\tfrac{1}{2} + \theta - \sqrt{\theta^2 - 1/4} > \frac{\sqrt{5}-1}{2} \\
&\iff 2\theta + 2 > \sqrt{5} + \sqrt{4\theta^2 - 1} \\
&\iff 4\theta > \sqrt{5}\sqrt{4\theta^2 - 1} \\
&\iff 5 > 4\theta^2.
\end{aligned}$$

‖ Vide *Die Lehre von den Kettenbrüchen*, O. Perron, 1929, p. 170.

Proof. If P_{v+1} and P_v are both greater than $\sqrt{R}$,

$$\begin{aligned}|P_{v+1} - P_v| &= (P_{v+1} - \sqrt{R}) - (P_v - \sqrt{R}) \\ &\leq \max(P_{v+1} - \sqrt{R}, P_v - \sqrt{R}) \\ &< Q_v \text{ by (F) §2.3}\end{aligned}$$

Similarly if P_{v+1} and P_v are both less than $\sqrt{R}$.

If $P_{v+1} > \sqrt{R}$ and $P_v < \sqrt{R}$, we have by (D) §2.3

$$R - (P_{v+1} - Q_v)^2 \geq P_{v+1}^2 - R \text{ and}$$
$$(P_v + Q_v)^2 - R \geq R - P_v^2.$$

Adding, we get

$$(P_v + P_{v+1})(P_v - P_{v+1} + 2Q_v \geq (P_{v+1} + P_v)(P_{v+1} - P_v),$$

so that $2(P_{v+1} - P_v) \leq 2Q_v$, i.e., $P_{v+1} - P_v \leq Q_v$.

If $P_{v+1} < \sqrt{R}$ and $P_v > \sqrt{R}$, we get in the same way, $P_v - P_{v+1} \leq Q_v$.

Equality will occur only when

$$(P_{v+1} - Q_v)^2 + P_{v+1}^2 = 2R = P_v^2 + (P_v + Q_v)^2,$$

i.e.,

$$P_{v+1}^2 - P_{v+1}Q_v + \frac{Q_v^2}{2} = R, \ (1)$$
$$P_v^2 + P_vQ_v + \frac{Q_v^2}{2} = R. \ (2)$$

Subtracting (2) from (1) gives

$$P_{v+1}^2 - P_v^2 - (P_{v+1} + P_v)Q_v = 0,$$

and hence

$$P_{v+1} - P_v = Q_v. \ (3)$$

We also have

$$P_{v+1} + P_v = b_vQ_v. \ (4)$$

(Adding (1) and (2) gives $P_{v+1}^2 + {P_v}^2 + Q_V(P_v - P_{v+1}) + Q_v^2 = 2R$ and hence (3) gives $P_{v+1}^2 + P_v^2 = 2R$, which is needed in the proof of Theorem IX.)

Hence

$$P_{v+1} = (b_v + 1)Q_v/2$$
$$P_v = (b_v - 1)Q_v/2.$$

Substituting in (2) gives $4R = Q_v^2(b_v^2 + 1)$, which implies that Q_v is even. Then (2) gives

$$\begin{aligned}\frac{R - P_v^2}{Q_v} &= P_v + \frac{Q_v}{2} \\ &= (b_v - 1)\frac{Q_v}{2} + \frac{Q_v}{2} \\ &= \tfrac{1}{2}b_vQ_v.\end{aligned}$$

Further, the surds being in standard form, $\frac{R-P_v^2}{Q_v}, P_v, Q_v$ have highest common factor unity, so common factor $\frac{Q_v}{2} = 1$.

Hence $Q_v = 2, R = b_v^2 + 1, P_v = b_v - 1, P_{v+1} = b_v + 1, Q_{v+1} = b_v, b_{v+1} = 1$ the latter following from $R = P_{v+1}^2 + b_{v+1}Q_{v+1}Q_v$, which gives $b_v = b_{v+1}Q_{v+1}$.

Now $b_v = 2$ would give $\frac{P_v+\sqrt{R}}{Q_v} = \frac{1+\sqrt{5}}{2}$, which is not semi-reduced. Hence $b_v > 2$.

Remark. The above proof only assumed that the first of the given surds is semi-reduced.

4. *Special Critical Fractions**

4.1.. In §2 of our previous communication† we have called the surds

(i) $\frac{1}{2} + \frac{\sqrt{R}}{Q} - \frac{\sqrt{4R-Q^2}}{2Q}$, $(|Q| < 2\sqrt{R})$,

(ii) $\frac{1}{2} + \frac{\sqrt{R}}{Q}$, $(|Q| > 2\sqrt{R})$,

critical fractions, since they decide the nature of the representations to be assigned to $\frac{P+\sqrt{R}}{Q}$ in a B.c.f. development. Ambiguities arise when

*This is a continuation of the memoir published in the *Journal of the Mysore University*, Vol. 1, part II, pp. 21–32.

†See *ibid.*, Vol. I, part II, p. 26.

(iii) $\frac{P+\sqrt{R}}{Q}-\frac{1}{2}-\frac{\sqrt{R}}{Q}+\frac{\sqrt{4R-Q^2}}{2Q}=\frac{P}{Q}-\frac{1}{2}+\frac{\sqrt{4R-Q^2}}{2Q}$ is an integer ($|Q|<2\sqrt{R}$), which implies $4R-Q^2=t^2$, where $\frac{2P+t}{Q}$ is an odd integer, Q and t are even integers and R is a sum of squares;

(iv) $\frac{P+\sqrt{R}}{Q}-\frac{1}{2}-\frac{\sqrt{R}}{Q}=\frac{P}{Q}-\frac{1}{2}$ is an integer ($|Q|>2\sqrt{R}$);

but these cases have been circumvented by appropriate conventions.

If $\frac{P+\sqrt{R}}{Q}$ be a special surd with $\frac{P_1+\sqrt{R}}{Q_1}$ as its successor, and $R=Q_1^2+\frac{1}{4}Q^2>Q^2+\frac{1}{4}Q_1^2$, then it is easily seen that the fractional part of $\frac{P+\sqrt{R}}{Q}$ in its positive representation is equal to the corresponding critical fraction which takes the special form $\frac{1}{2}+\frac{\sqrt{R}-Q_1}{Q}$, where $Q_1>|Q|$.

Definition. A proper fraction of the form $\frac{q-p+\sqrt{p^2+q^2}}{2q}$ is called a *special critical fraction* when $p>2q>0$.

4.2. Theorem VIII. If $\frac{P_{v-1}+\sqrt{R}}{Q_{v-1}}$ is a special surd with successors $\frac{P_v+\sqrt{R}}{Q_v}$, $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}}$, $\frac{P_{v+2}+\sqrt{R}}{Q_{v+2}}$, then $\frac{P_{v+1}+\sqrt{R}}{Q_v}$ is a successor of $\frac{P_{v+2}+\sqrt{R}}{Q_{v+1}}$ (with $\frac{P_{v+2}+\sqrt{R}}{Q_{v+1}}=b_{v+1}+\frac{\epsilon_{v+1}Q_v}{P_{v+1}+\sqrt{R}}$) in all cases except when $R=Q_v^2+\frac{1}{4}Q_{v-1}^2$. In this case, $\xi_v=(1-g)^{-1}$, where g is a special critical fraction and $\epsilon_v=-1, \epsilon_{v+1}=1$.

Proof. Let

$$\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}}=b_{v+1}+\frac{\epsilon_{v+2}Q_{v+2}}{P_{v+2}+\sqrt{R}}.$$

$$\text{Then }\frac{P_{v+1}-\sqrt{R}}{Q_{v+1}}=b_{v+1}-\frac{\epsilon_{v+2}Q_{v+2}}{P_{v+2}-\sqrt{R}},$$

$$\text{i.e., }-\frac{\epsilon_{v+1}Q_{v+1}}{P_{v+1}+\sqrt{R}}=b_{v+1}-\frac{P_{v+2}+\sqrt{R}}{Q_{v+1}}.$$

$$\text{Hence }\frac{P_{v+2}+\sqrt{R}}{Q_{v+1}}=b_{v+1}+\frac{\epsilon_{v+1}Q_v}{P_{v+1}+\sqrt{R}}. \qquad (1)$$

Now by (C), (1) is a Bhaskara representation if $Q_v^2+\frac{1}{4}Q_{v+1}^2<R$ or if $Q_v^2+\frac{1}{4}Q_{v+1}^2=R$ and $\epsilon_{v+1}=-1$. So we assume that

$$Q_v^2+\tfrac{1}{4}Q_{v+1}^2=R \text{ and } \epsilon_{v+1}=1.$$

We now show that this is equivalent to the single condition $Q_v^2+\frac{1}{4}Q_{v-1}^2=R$.

Similar to (1), we have

$$\frac{P_{v+1}+\sqrt{R}}{Q_v} = b_v + \frac{\epsilon_v Q_{v-1}}{P_v+\sqrt{R}}. \qquad (2)$$

We also have $Q_{v-1}^2 + \frac{1}{4}Q_v^2 \le R$ and $Q_v^2 + \frac{1}{4}Q_{v-1}^2 \le R$.

Hence

$$\begin{aligned} P_{v+1}^2 &= R - Q_vQ_{v+1} \\ &= Q_v^2 + \tfrac{1}{4}Q_{v+1}^2 - Q_vQ_{v+1} \\ &= (Q_v - \tfrac{1}{2}Q_{v+1})^2. \qquad (2') \end{aligned}$$

But since $Q_v^2 + \frac{1}{4}Q_{v+1}^2 = R \ge Q_{v+1}^2 + \frac{1}{4}Q_v^2$, we have $Q_v \ge Q_{v+1}$. Then (2') gives

$$P_{v+1} = Q_v - \tfrac{1}{2}Q_{v+1}. \qquad (3)$$

Hence

$$\begin{aligned} \frac{P_{v+1}+\sqrt{R}}{Q_v} &= \frac{\sqrt{R}+Q_v-\frac{1}{2}Q_{v+1}}{Q_v} \\ &= 2 + \frac{\sqrt{R}-Q_v-\frac{1}{2}Q_{v+1}}{Q_v} \\ &= 2 - \frac{Q_{v+1}}{\sqrt{R}+Q_v+\frac{1}{2}Q_{v+1}}. \qquad (4) \end{aligned}$$

Comparing (2) and (4) gives $b_v = 2, -\epsilon_v Q_{v-1} = Q_{v+1}, P_v - Q_v = \frac{1}{2}Q_{v+1}$.

Hence

$$\tfrac{1}{4}Q_{v-1}^2 + Q_v^2 = \tfrac{1}{4}Q_{v+1}^2 + Q_v^2 = R.$$

Conversely if $\frac{1}{4}Q_{v-1}^2 + Q_v^2 = R$, we see $Q_v \ge |Q_{v-1}|$. Then

$$P_v = Q_v - \tfrac{1}{2}Q_{v-1}.$$

Also because $\frac{P_{v-1}+\sqrt{R}}{Q_{v-1}} = b_{v-1} + \frac{\epsilon_v Q_v}{P_v+\sqrt{R}}$ is a Bhaskara represention with ambiguous case, we must have $\epsilon_v Q_{v-1} < 0$. Hence $P_v = Q_v + \frac{1}{2}|Q_{v-1}|$ and

$$\begin{aligned} \frac{P_v+\sqrt{R}}{Q_v} &= 2 + \frac{\sqrt{R}+\frac{1}{2}|Q_{v-1}|-Q_v}{Q_v} \\ &= 2 + \frac{|Q_{v-1}|}{\sqrt{R}-\frac{1}{2}|Q_{v-1}|+Q_v}, \end{aligned}$$

so that $\epsilon_{v+1} = 1, Q_{v+1} = |Q_{v-1}|$ and $Q_v^2 + \frac{1}{4}Q_{v+1}^2 = R$.

Consequently (1) implies $\frac{P_{v+1}+\sqrt{R}}{Q_v}$ will fail to be a successor of $\frac{P_{v+2}+\sqrt{R}}{Q_{v+1}}$.

4.3. Theorem IX. Two different semi-reduced surds cannot have the same Bhaskara successor, unless they are conjugates of $-g$ and $1-g$, where g is a special critical fraction.

Proof. Let two different semi-reduced surds $\xi_v = \frac{P_v+\sqrt{R}}{Q_v}$ and $\xi_v' = \frac{P_v'+\sqrt{R}}{Q_v'}$ have the same successor ξ_{v+1}. Then $P_v' \neq P_v$. For

$$R = P_v^2 + \epsilon_{v+1}Q_vQ_{v+1} = P_v'^2 + \epsilon_{v+1}'Q_v'Q_{v+1}.$$

Then $P_v = P_v'$ implies $\epsilon_{v+1}Q_v = \epsilon_{v+1}'Q_v'$ and since Q_v and Q_v' are positive, being semi-reduced, we would have $Q_v = Q_v'$.

Hence we can assume $P_v' > P_v$.

Now the following are Bhaskara representations:

$$\frac{P_v + \sqrt{R}}{Q_v} = b_v + \frac{\epsilon_{v+1}Q_{v+1}}{P_{v+1} + \sqrt{R}}, \text{ and } \frac{P_v' + \sqrt{R}}{Q_v'} = b_v' + \frac{\epsilon_{v+1}'Q_{v+1}}{P_{v+1} + \sqrt{R}}.$$

Hence

$$\frac{P_v + \sqrt{R}}{Q_v} + t\frac{P_v' + \sqrt{R}}{Q_v'} \in \mathbb{Z}, \qquad (1).$$

where $t = \epsilon_{v+1}\epsilon_{v+1}' = \pm 1$. Hence $Q_v = Q_v'$ and $t = -1$.

Then

$$P_v - P_v' \,(\operatorname{mod} Q_v). \qquad (2)$$

Arguing as in the proof of Theorem VII*, replacing P_{v+1} by P_v', but omitting the consideration $P_v' < \sqrt{R}, P_v > \sqrt{R}$, which is not true here, we get

$$P_v' - P_v \leq Q_v. \qquad (3)$$

From (2) and (3), $P_v' = P_v$ or $P_v' - P_v = Q_v$ and in the latter case $P_v'^2 + P_v^2 = 2R$, from which we derive $P_v = |Q_{v-1}| - \frac{1}{2}Q_v$.

(For

$$(P_v + Q_v)^2 + P_v^2 = 2R = 2P_v^2 + 2\epsilon_vQ_vQ_{v-1}$$
$$(P_v + Q_v)^2 = P_v^2 + 2\epsilon_vQ_vQ_{v-1}$$

*See *Journal of the Mysore University*, Vol. I, Part II, page 31.

so $\epsilon_v Q_{v-1} > 0$, i.e., $\epsilon_v Q_{v-1} = |Q_{v-1}|$.

Hence

$$\begin{aligned}(P_v + Q_v)^2 &= P_v^2 + 2Q_v|Q_{v-1}|\\ 2P_vQ_v + Q_v^2 &= 2Q_v|Q_{v-1}|\\ 2P_v + Q_v &= 2|Q_{v-1}|.\end{aligned}$$

Hence $P_v = |Q_{v-1}| - \frac{1}{2}Q_v$.)

Then $P_v' = P_v + Q_v = |Q_{v-1}| + \frac{1}{2}Q_v$.

Also

$$\begin{aligned}R &= P_v^2 + |Q_{v+1}|Q_v\\ &= (|Q_{v-1}| - \tfrac{1}{2}Q_v)^2 + |Q_{v+1}|Q_v\\ &= Q_{v-1}^2 + \tfrac{1}{4}Q_v^2.\end{aligned}$$

Thus the two surds which have the same successor are of the form

$$\xi_v = \frac{|Q_{v-1}| - \frac{1}{2}Q_v + \sqrt{R}}{Q_v}, \xi_v' = \frac{|Q_{v-1}| + \frac{1}{2}Q_v + \sqrt{R}}{Q_v} = 1 + \xi_v,$$

where $R = Q_{v-1}^2 + \frac{1}{4}{Q_v}^2 \geq {Q_v}^2 + \frac{1}{4}{Q_{v-1}}^2$ by Theorem VIII, so $Q_v < |Q_{v-1}|$. Also Q_v is even.

Obviously $\frac{\frac{1}{2}Q_v - |Q_{v-1}| + \sqrt{R}}{Q_v}$ is a special critical fraction, g say. Then ξ_v is the conjugate of $-g$ and ξ_v' is the conjugate of $1 - g$,

4.4. Theorem X. If g be a special critical fraction, then

(i) g^{-1} has no Bhaskara predecessor,

(ii) $(1 - g)^{-1}$ is semi-reduced,

(iii) the Bhaskara successors of g^{-1} and $(1 - g)^{-1}$ are respectively the conjugates of $1 - g$ and $-g$;

(iv) the conjugate of $1 - g$ has no semi-reduced predecessor,

(v) the conjugate of $-g$ has a unique semi-reduced predecessor,

Proof. Let $g = \frac{q-p+\sqrt{p^2+q^2}}{2q}, p > 2q > 0$. Then a predecessor of g^{-1} or $(1-g)^{-1}$ will be of the form $a \pm g$, where a is an integer.

Put $\frac{P+\sqrt{R}}{Q} = a + g = a + \frac{p}{p-q+\sqrt{R}} = a + 1 - (1-g) = a + 1 - \frac{p}{p+q+\sqrt{R}}$, where $R = p^2 + q^2$.

Then $Q = 2q < p < \sqrt{R}, \quad p^2 + \frac{1}{4}Q^2 = R > Q^2 + \frac{1}{4}p^2$, so $\frac{P+\sqrt{R}}{Q}$ is a special surd.

Hence g^{-1} has no predecessor of the form $a + g$, while $(1-g)^{-1}$ has one of the form $a + 1 - (1-g)$.

Similarly, it can be shown that g^{-1} has no predecessor of the form $a - g$, while $(1-g)^{-1}$ has one of the form $a + 1 + (1-g)$.

Now

$$\begin{aligned} g^{-1} &= \frac{p-q+\sqrt{R}}{p} = 1 + \frac{p}{q+\sqrt{R}} = 2 - \frac{2q}{p+q+\sqrt{R}} \\ &= 2 - \frac{1}{\text{conjugate of } (1-g)}; \end{aligned}$$

and

$$\begin{aligned} (1-g)^{-1} &= \frac{p+q+\sqrt{R}}{p} = 3 - \frac{3p-4q}{2p-q+\sqrt{R}} = 2 + \frac{2q}{p-q+\sqrt{R}} \\ &= 2 + \frac{1}{\text{conjugate of } -g}. \end{aligned}$$

Since $2q < p < 3p - 4q$, the Bhaskara successors of g^{-1} and $(1-g)^{-1}$ are respectively the conjugates of $1-g$ and $-g$.

Any predecessor of the conjugate of $1-g$ must be of the form $a \pm \frac{p+q-\sqrt{R}}{p}$, where a is an integer. For a semi-reduced predecessor, $a + \frac{p+q-\sqrt{R}}{p}$ is inadmissible and a must be an integer such that $p(a-1) - q > 0$ and $(pa - p - q)^2$ is nearest to R; all these conditions are satisfied only when $a = 2$, for it can be easily verified that $p - q < \sqrt{R}, pa - p - q > \sqrt{R}$ when $a > 2$ and $R - (p-q)^2 < (2p-q)^2 - R$, when $p > 2q$. Thus the only possible semi-reduced predecessor of the conjugate of $1-g$ is g^{-1}. But since g^{-1} has no Bhaskara predecessor, it cannot be semi-reduced.

Similarly, the possible semi-reduced predecessors of the conjugate of $-g$ must be of the form $\frac{pa-p+q+\sqrt{R}}{p}$, where a is an integer such that $pa-p+q>0$ and $(pa-p+q)^2$ is nearest to R. Obviously $a=2$, since when $a\geq 2, pa-p+q>\sqrt{R}$ and when $a=1, q<\sqrt{R}$, while $(p+q)^2-R<R-q^2$. Thus the possible semi-reduced predecessor is $(1-g)^{-1}$, which is certainly semi-reduced, with a special surd as its predecessor.

Corollary 1. Two different reduced surds cannot have the same successor.

From the above proof, we see that the following is true:

Corollary 2[4]. Neither the conjugate of $-g$ nor that of $1-g$ can be the successor of a standard surd of the form $\frac{\sqrt{R}}{Q}$.

5. *Pure Recurring Bhaskara Continued Fractions*

5.1. Definition. A pure recurring B.c.f. is one in which the complete quotients recur from the first.

We have already seen that the complete quotients in a B.c.f. development are ultimately reduced surds. Hence a pure recurring B.c.f. is a reduced surd.

The converse of this will now be proved.

5.2. Theorem XI. The Bhaskara development of a reduced surd is a pure recurring half-regular continued fraction.

Proof. Let $\xi_0 = \frac{P_0+\sqrt{R}}{Q_0}$ be a reduced surd and let its B.c.f. development be

$$\xi_0 = b_0 + \frac{\epsilon_1|}{|b_1} + \cdots + \frac{\epsilon_{k-1}|}{|b_{k-1}} + \frac{\epsilon_k|}{|\underset{*}{b_k}} + \cdots + \frac{\epsilon_{k+n-1}|}{|\underset{*}{b_{k+n-1}}},$$

where $\xi_{k+v} = \xi_{k+v+tn}$, $(v=0,1,\ldots,n-1)$, $t\geq 1$ and $b_{k+v}=b_{k+v+tn}$.

Since ξ_0 is reduced, ξ_{k-1} and ξ_{k+n-1} are also reduced; but their respective successors ξ_k and ξ_{k+n} are equal.

By Theorem X, Corollary 1, therefore $\xi_{k-1}=\xi_{k+n-1}$.

If $\epsilon_{k-1}\neq\epsilon_{k+n-1}$, then $\xi_{k-2}\neq\xi_{k+n-2}$, which will contradict Theorem X, Corollary (1), so that $\epsilon_{k-1}=\epsilon_{k+n-1}$, i.e., the recurrence begins one step earlier. This process can be evidently continued backwards until ξ_0 is reached.

[4]Not quite! Rather an exercise

The first complete quotient therefore recurs and the h.r.c.f. is pure recurring one, of the form $\xi_0 = \underset{*}{b_0} + \frac{\epsilon_1|}{|b_1} + \cdots + \frac{\epsilon_{n-1}|}{|\underset{*}{b_{n-1}}}$.

5.3. Theorem XII. The B.c.f. development of the standard surd $\frac{\sqrt{R}}{Q}(>1)$ has only one term in its acyclic part.

Proof. Let $\xi_0 = \frac{\sqrt{R}}{Q} = b_0 + \frac{\epsilon_1}{\xi_1}$ be a B.c.f., where $\xi_1 = \frac{P_1+\sqrt{R}}{Q_1}$. Then

$$P_1 = b_0 Q, \quad \epsilon_1 Q Q_1 = R - P_1^2.$$

$\frac{\sqrt{R}}{Q}$ being in standard form, we may write $R = QQ'$, where $\gcd(Q, Q') = 1$; hence $\epsilon_1 Q_1 = Q' - b_0^2 Q$.

By Theorem I, since $Q < \sqrt{R}$, we have $P_1 > 0, Q_1 > 0$ and

$$|Q_1 - \tfrac{1}{2}\epsilon_1 Q| \le P_1. \qquad (1)$$

We now prove $|Q - \frac{1}{2}\epsilon_1 Q_1| \le P_1$.

Case 1. $Q < \frac{1}{2}Q_1$ and $\epsilon_1 = 1$. Then

$$|Q - \tfrac{1}{2}\epsilon_1 Q_1| = \tfrac{1}{2}Q_1 - Q < Q_1 - \tfrac{1}{2}Q \le P_1 \text{ by (1).}$$

Case 2. Assume $Q \ge \frac{1}{2}Q_1$ and $\epsilon_1 = 1$. Then we have to prove

$$Q - \tfrac{1}{2}Q_1 \le P_1,$$
$$\text{i.e., } Q - \tfrac{1}{2}Q' + \tfrac{1}{2}b_0^2 Q \le b_0 Q$$
$$\text{i.e., } Q(1 - b_0 + \tfrac{1}{2}b_0^2) \le \tfrac{1}{2}Q',$$
$$\text{i.e., } (b_0 - 1)^2 \le \frac{Q'}{Q} - 1. \qquad (2)$$

As $\epsilon_1 = 1$, we have $P_1 < \sqrt{R}$, i.e., $b_0 Q < \sqrt{QQ'}$. Hence $b_0^2 < \frac{Q'}{Q}$, so that

$$(b_0 - 1)^2 \le b_0^2 - 1 < \frac{Q'}{Q} - 1$$

and (2) holds.

Case 3. Assume $\epsilon_1 = -1$. Then from (1), we have $Q_1 + \frac{1}{2}Q \le P_1$ and hence

$$b_0^2 Q - Q' + \tfrac{1}{2}Q \le b_0 Q$$
$$b_0^2 - b_0 + \tfrac{1}{2} \le \frac{Q'}{Q}.$$

Also since $\epsilon_1 = -1$, we are dealing with the negative representation of ξ_0, so $b_0 > 1$. Hence $(b_0 - 1)^2 + 1 < b_0^2 - b_0 + \frac{1}{2}$; hence (2) again holds.

Thus in all cases,

$$|Q - \tfrac{1}{2}\epsilon_1 Q_1| \leq P_1. \qquad (5)$$

holds.

From, (1) and (5), $\frac{\sqrt{R}}{Q}$ is a special surd and therefore ξ_1 is a semi-reduced surd, ξ_2 is a reduced surd and the period of recurrence must begin at least from ξ_2, the successor of ξ_1.

By Theorem X, Cor. 2, ξ_1 cannot be the conjugate of $-g$ or $1-g$, where g is a special critical fraction. Therefore ξ_1 is the unique semi-reduced predecessor of ξ_2. Hence ξ_1 must recur.

Further ξ_0 cannot recur. For if $\xi_0 = \xi_{n+1}$, with $n > 0$, then $P_{n+1} = 0$ and $Q_n Q_{n+1} = R$, an impossible relation when each of Q_n, Q_{n+1} is less than $\sqrt{R}$.

Hence the recurring period begins from ξ_1 and the B.c.f. development of $\frac{\sqrt{R}}{Q}$ has one and only one term in the acyclic part.

Corollary. b_0 is such that $b_0^2 Q^2$ is the nearest to R among the aquare multiples of Q^2.

5.4. Theorem XIII. If g be a special critical function, then $(1-g)^{-1}$ develops as pure recurring B.c.f.

Proof. We know that $(1-g)^{-1}$ is of the form $\frac{p+q+\sqrt{R}}{p}$, where $p > 2q > 0$ and $R = p^2 + q^2$. It is sufficient for our purpose to prove that there exists a Bhaskara predecessor of $(1-g)^{-1}$ which is semi-reduced and the rest will follow from Theorem XI.

As we have seen already in Theorem X, a semi-reduced predecessor of $(1-g)^{-1}$ must be of the form $\frac{(2n-1)q-p+\sqrt{R}}{2q}$, where $n \geq 2$ and $(2n-1)q-p > 0$. Also its Bhaskara predecessor is a special surd of the form

$$\mu + \frac{2q\epsilon}{(2n-1)q - p + \sqrt{R}} = \mu - \frac{p - (2n-1)q + \sqrt{R}}{\epsilon\{(2n^2 - 2n)q - p(2n-1)\}}, \qquad (0)$$

where μ is an integer and $\epsilon = \pm 1$. (AAK has a $+$ sign instead of a $-$ sign on the RHS of (0), but (1) and (2) below follow from the $-$ sign.)

The condition for special surds becomes

$$|2q - \tfrac{1}{2}(2n-1)p + q(n^2-n)| \le (2n-1)q - p, \qquad (1)$$
$$|q - (2n-1)p + q(2n^2-2n)| \le (2n-1)q - p. \qquad (2)$$

We have to consider four cases:

(i) $2q - \frac{1}{2}(2n-1)p + q(n^2-n) \ge 0,\ \ q - (2n-1)p + q(2n^2-2n) \ge 0$;
(ii) $2q - \frac{1}{2}(2n-1)p + q(n^2-n) \ge 0,\ \ q - (2n-1)p + q(2n^2-2n) < 0$;
(iii) $2q - \frac{1}{2}(2n-1)p + q(n^2-n) < 0,\ \ q - (2n-1)p + q(2n^2-2n) \ge 0$;
(iv) $2q - \frac{1}{2}(2n-1)p + q(n^2-n) < 0,\ \ q - (2n-1)p + q(2n^2-2n) < 0$.

Case (iii) is impossible.

(i) is equivalent to $p/q \le \frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}}$ and $p/q \le \frac{n^2-n+2}{n-\frac{1}{2}}$.

(ii) is equivalent to $\frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}} < p/q \le \frac{n^2-n+2}{n-\frac{1}{2}}$.

(iv) is equivalent to $\frac{n^2-n+2}{n-\frac{1}{2}} < p/q$ and $\frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}} < p/q$.

This corresponds to p/q belonging to one of the intervals

$(2, \frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}}], (\frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}}, \frac{n^2-n+2}{n-\frac{1}{2}}], (\frac{n^2-n+2}{n-\frac{1}{2}}, \infty)$.

Also in case (i),

(1) and (2) become $\frac{n^2-3n+3}{n-\frac{3}{2}} \le p/q$ and $n-1 \le p/q$.

In case (ii),

(1) and (2) become $\frac{n^2-3n+3}{n-\frac{3}{2}} \le p/q$ and $p/q \le n$.

Finally, in case (iv),

(1) and (2) become $p/q \le \frac{n^2+n+1}{n+\frac{1}{2}}$ and $p/q \le n$.

But if $m > 4, m \in \mathbb{N}$, the intervals

(a) $(m-1, \frac{m^2-m+\frac{1}{2}}{m-\frac{1}{2}}]$, (b) $(\frac{m^2-m+\frac{1}{2}}{m-\frac{1}{2}}, \frac{m^2-m+2}{m-\frac{1}{2}}]$, (c) $(\frac{m^2-m+2}{m-\frac{1}{2}}, m]$

divide up the interval $p/q > 4$.

So if $p/q > 4$, let n be the unique integer $m > 4$ such that one of intervals (a), (b) and (c) contains p/q.

In case (a), then (i) holds and as $\frac{n^2-3n+3}{n-\frac{3}{2}} \le n-1$, (1) and (2) are satisfied.

In case (b), then (ii) holds and as $\frac{n^2-3n+2}{n-\frac{3}{2}} \le \frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}}$ and $\frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}} \le n$, (1) and (2) are satisfied.

In case (c), then (iv) holds and as $n \le \frac{n^2+n+1}{n+\frac{1}{2}}$, (1) and (2) are satisfied.

Finally, if $n = 3$ and $2 < p/q \le 13/5$ (case (i)) or $13/5 < p/q \le 3$ (case(ii)), then (1) and (2) hold; while if $n = 4$ and $3 < p/q \le 25/7$ (case (i)) or $25/7 < p/q \le 4$ (case(ii)), then again (1) and (2) hold.

In fact n is the least integer exceeding p/q, if q does not divide p and p/q otherwise.

5.5. Before discussing further the properties of the recurring B.c.f., we require certain lemmas on the behaviour of unit partial quotients in simple continued fractions.

Lemma 1. If $\xi = \frac{P+\sqrt{R}}{Q}$ develops as a pure recurring simple continued fraction, with a sequence of at least three successive unit partial quotients preceded and followed by other partial quotients, then the denominators of the complete quotients corresponding to the unit partial quotients, other than the first and last of the sequence, are less than $\sqrt{R}$.

With

$$\xi = \frac{P+\sqrt{R}}{Q} = \underset{*}{a_0} + \frac{1|}{|a_1} + \cdots + + \frac{1|}{|a_r} + \frac{1|}{|1} + \cdots + \frac{1|}{|1} + \frac{1|}{|a_{r+n+1}} + \cdots + \frac{1|}{|\underset{*}{a_p}},$$

$$= (\underset{*}{a_0}, a_1, \ldots, a_r, 1_{[n]}, a_{r+n+1}, \ldots, \underset{*}{a_p}).$$

$$\begin{aligned}\xi_{r+v} &= \frac{P_{r+v}+\sqrt{R}}{Q_{r+v}} \\ &= (\underset{*}{1_{[n-v+1]}}, a_{n+r+1}, \ldots, a_p, a_0, a_1, \ldots, a_r, \underset{*}{1_{[v-1]}}) = f, \text{ say} \qquad (1)\end{aligned}$$

By* Galois' theorem of inverse periods,

$$\frac{-Q_{r+v}}{P_{r+v}-\sqrt{R}} = (\underset{*}{1_{[v-1]}}, a_r, \ldots, a_0, a_p, \ldots, a_{n+r+1}, \underset{*}{1_{[n-v+1]}}).$$

Hence

$$\frac{-P_{r+v}+\sqrt{R}}{Q_{r+v}} = (0, \underset{*}{1_{[v-1]}}, a_r, \ldots, a_0, a_p, \ldots, a_{n+r+1}, \underset{*}{1_{[n-v+1]}}) = f' \text{ (say)} \quad (2).$$

*Vide pp. 82-85, *Die Lehre von den Kettenbrüchen*, O. Perron, 1929.

Adding (1) and (2), $\frac{2\sqrt{R}}{Q_{r+v}} = f + f'$, so that $Q_{r+v} \lessgtr \sqrt{R}$, according as $f \gtrless 2 - f' = f''$ (say).

But

$$\begin{aligned} 2 - f' &= 2 - (0, \underset{*}{1_{[v-1]}}, a_r, \ldots, a_0, a_p, \ldots, a_{n+r+1}, \underset{*}{1_{[n-v+1]}}) \\ &= (1, 1, 0, 1_{[v-2]}, \ldots) \\ &= (1, 2, 1_{[v-3]}, \ldots). \end{aligned}$$

If $n-1 \geq v \geq 3$, the second complete quotient of f is less than the corresponding complete quotient of f'' and therefore $f > f''$, implying $Q_{r+v} < \sqrt{R}$.

If $v = 2$ and $n \geq 3$, we have

$$\begin{aligned} f'' = 2 - (0, 1, a_r, \ldots) &= (1, 1 + a_r, \ldots) \\ &< 1 + \frac{1}{1 + a_r} \leq 1 + \frac{1}{2}. \end{aligned}$$

Also $f = (1, 1, \xi), \xi > 1$, so $f \geq 1 + \frac{1}{2}$. Hence $f'' < f$ and again $Q_{r+v} < \sqrt{R}$.

Thus for all values of v greater than 1 and less than n (≥ 3), $Q_{r+v} < \sqrt{R}$.

Lemma 1.1. Let $n > 2$.

(i) If $a_r > 2$, then $Q_{r+1} > \sqrt{R}$;

(ii) If $a_{r+n+1} > 2$, then $Q_{r+n} > \sqrt{R}$.

Proof. $f = (1, 1, 1, \xi), \xi > 1$.

(i) Then $f = \frac{3\xi+2}{2\xi+1} < \frac{5}{3}$.

Also $f' < \frac{1}{a_r} \leq \frac{1}{3}$ if $a_r > 2$. Hence $f + f' < 2$ in this case.

(ii) Now assume $a_{r+n+1} > 2$. Then

$f = (1, a_{n+r+1}, \ldots)$ and $f' = (1, 1, \xi), \xi > 1$; so $f < 1 + \frac{1}{a_{n+r+1}} \leq \frac{4}{3}$. Also $f' = \frac{\xi+1}{2\xi+1} < \frac{2}{3}$. Hence $f + f' < 2$.

Similarly, we have

Lemma 1.2. Let $n = 2$.

(i) If $a_r - 1 < a_{r+n+1}$, then $Q_{r+1} < \sqrt{R}$, while if $a_r + 1 < a_{r+n+1}$, then $Q_{r+n} > \sqrt{R}$;

(ii) if $a_{r+n+1}+1 < a_r$, then $Q_{r+1} > \sqrt{R}$, while if $a_{r+n+1}-1 < a_r$, $Q_{r+n} < \sqrt{R}$;

(iii) if $a_r = a_{r+n+1}$, then $Q_{r+1} < \sqrt{R}$ and $Q_{r+n} < \sqrt{R}$.

Proof.
Case 1. $n = 2, v = 1$. Then
$f = (1, 1, a_{r+3}, \ldots), f' = (0, a_r, \ldots)$ and hence

$$1 + \frac{1}{1+\frac{1}{b}} < f < 1 + \frac{1}{1+\frac{1}{b+1}}$$
$$\frac{1}{1+a} < f' < \frac{1}{a},$$

where $a = a_r$ and $b = a_{r+3}$.

So $f + f' < 1 + \frac{b+1}{b+2} + \frac{1}{a}$. Hence $Q_{r+3} > \sqrt{R}$ if $1 + \frac{b+1}{b+2} + \frac{1}{a} \leq 2$ and this reduces to $b + 1 < a$.

Also $f + f' > 1 + \frac{b}{b+1} + \frac{1}{1+a}$. Hence $Q_{r+3} < \sqrt{R}$ if $1 + \frac{b}{b+1} + \frac{1}{1+a} \geq 2$ and this reduces to $b > a - 1$.

Case 2. $n = 2, v = 2$. Then $f = (1, a_{r+3}, \ldots), f' = (0, 1, a_r, \ldots)$ and hence

$$1 + \frac{1}{b+1} < f < 1 + \frac{1}{b}$$
$$\frac{1}{1+\frac{1}{a}} < f' < \frac{1}{1+\frac{1}{a+1}},$$

where $a = a_r$ and $b = a_{r+3}$.

So, $f + f' < 1 + \frac{1}{b} + \frac{a+1}{a+2}$. Hence
$Q_{r+3} > \sqrt{R}$ if $1 + \frac{1}{b} + \frac{a+1}{a+2} \leq 2$ and this reduces to $a + 2 \leq b$ or $a + 1 < b$.

Also $f + f' > 1 + \frac{1}{b+1} + \frac{a}{a+1}$. Hence $Q_{r+3} < \sqrt{R}$ if $1 + \frac{1}{b+1} + \frac{a}{a+1} \geq 2$ and this reduces to $a \geq b$ or $a + 1 > b$.

Lemma 2. In the simple continued fraction development of a surd of the form $\frac{q+\sqrt{p^2+q^2}}{p}$, p, q being integers such that $p > 2q > 0$, there cannot occur a complete quotient of the same form more than once in the recurring period; when such a complete quotient does occur, the recurring period is symmetric, with an even number of terms, which include a central sequence of an even number (possibly zero) of unit partial quotients.

Proof. Let $\xi_0 = \frac{P_0+\sqrt{R}}{Q_0} = \frac{q+\sqrt{p^2+q^2}}{p}$, where $p > 2q > 0$. Let $\xi_v = \frac{P_v+\sqrt{R}}{Q_v}$ be the v-th successor of ξ_0. Let $\overline{\xi_0}$ be the conjugate of ξ_0. Then $\xi_0\overline{\xi_0} = -1$. Also

$$1 < \xi_0 < \frac{1+\sqrt{5}}{2} = (1, 1, \ldots) = (\underset{*}{1}) = (1_\infty) \text{ and } -1 < \overline{\xi_0} < 0. \qquad (1)$$

By a well-known theorem of Galois, as ξ_0 is a reduced quadratic surd, the simple continued fraction for ξ_0 has a purely recurring period $(\underset{*}{a_0}, a_1, \ldots, \underset{*}{a_n})$ say.

From (1), $a_0 = 1$ and if a_m is the first partial quotient greater than 1, m must be odd; for if m be even, we have successively

$$\begin{aligned} (a_m, \ldots) &> (1_\infty) \\ (1, a_m, \ldots) &< (1_\infty) \\ (1_{[2]}, a_m, \ldots) &> (1_\infty) \\ &\cdots\cdots \\ (1_{[m]}, a_m, \ldots) &> (1_\infty), \end{aligned}$$

which contradicts (1). Hence

$$\xi_0 = (\underset{*}{1_{[m]}}, a_m, \ldots, \underset{*}{a_n}). \qquad (2)$$

$$\text{Also, } \xi_0 = -1/\overline{\xi_0} = (\underset{*}{a_n}, \ldots, \underset{*}{1_{[m]}}). \qquad (3)$$

Comparing (2) and (3), we have $a_n = a_{n-1} = \cdots = a_{n-m+1} = 1, a_m = a_{n-m}$: i.e., the period is a symmetric one, beginning and ending with an odd number of unit partial quotients.

Comparison of the complete quotients in (2) and (3) gives

$$\frac{P_v + \sqrt{R}}{Q_v} = \frac{P_{n+1-v} + \sqrt{R}}{Q_{n-v}}, (v \leq n),$$

i.e., $P_v = P_{n-v+1}, Q_v = Q_{n-v}$.

If $Q_v = Q_{v-1}$, then

$$\begin{aligned} \xi_v = \frac{P_v + \sqrt{R}}{Q_v} &= \frac{P_{n+1-v} + \sqrt{R}}{Q_{v-1}} \\ &= \frac{P_{n+1-v} + \sqrt{R}}{Q_{n+1-v}} = \xi_{n+1-v}. \end{aligned}$$

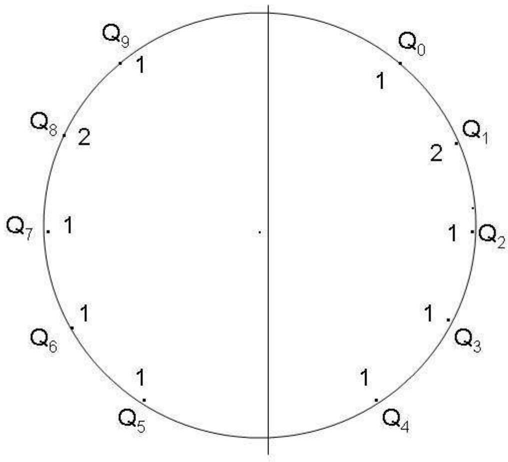

Figure 1: Partial quotients for rcf of $\frac{27+\sqrt{7453}}{82}$ arranged symmetrically

and so $v = n+1-v$, i.e., $v = (n+1)/2$, which implies that n should be odd.

Thus, only when n is odd, $Q_{\frac{n+1}{2}} = Q_{\frac{n-1}{2}}$ and these are the only consecutive $Q's$ which can be equal to each other. (4)

If a complete quotient, say, ξ_v, should be of the same form as ξ_0, its simple continued fraction development should have the same properties.

Writing $Q_0, Q_1, \ldots, Q_n$ around a circle at the vertices of a regular polygon of $n+1$ sides, we find that that they arrange themselves symmetrically about a diameter, such that the Q's symmetrically placed about this diameter are also equal, since $Q_v = Q_{n-v}$. (See figure 1 above.)

The symmetry of the Q's corresponding to ξ_v imply that $Q_v = Q_{v-1}$, just as $Q_0 = Q_n$. From (4) we see that this can happen only once and so, there cannot be more than one ξ_v of the same form as ξ_0 and it occurs when n is odd and $v = \frac{n+1}{2}$. In this case, we realise that same symmetry of Q's starting from $Q_{\frac{n+1}{2}}$, going round the circle and ending with $Q_{\frac{n-1}{2}}$ as in the first set $Q_0, Q_1, \ldots, Q_n$.

Thus we see that there exists a complete quotient ξ_v of the same form as ξ_0, only when n is odd and $Q_{\frac{n+r}{2}} = Q_{\frac{n-r}{2}}$, $r = 1, 3, 5, \ldots, n$.

Hence, if ξ_0 should have a remote successor of the same form as itself in the recurring period of its simple continued fraction development, then the recurring period must consist of an even number of symmetrically disposed partial quotients, including an initial, a central and a final set of unit partial quotients. In order that the recurring cycle may not lose its character as a primitive period, it is necessary that the first half of the cycle is not itself symmetical.

Example. $\frac{27+\sqrt{27^2+82^2}}{82} = (\underset{*}{1}, 2, 1, 1, 1, 1, 1, 1, 2, \underset{*}{1})$ has a remote successor $\frac{P_5+\sqrt{P_5^2+Q_5^2}}{Q_5}$ within the recurring period of the same form $\frac{37+\sqrt{37^2+78^2}}{78}$.

Also $\frac{37+\sqrt{37^2+78^2}}{78} = (\underset{*}{1}, 1, 1, 2, 1, 1, 2, 1, 1, \underset{*}{1})$

Lemma 3. If the standard surd of the form $\frac{\sqrt{R}}{Q_0}$ have in its simple continued fraction development, a complete quotient of the form $\frac{q+\sqrt{p^2+q^2}}{p}$, where $p > 2q > 0$, then the symmetric portion of the recurring period of partial quotients will include a central even number of the form $4n-2$ of unit partial quotients; also there cannot occur any other complete quotient of similar form within the recurring period, which must consist of an odd number of terms.

Conversely, if any simple continued fraction development of the standard surd $\frac{\sqrt{R}}{Q_0}$ has in its recurring period an odd number of partial quotients with a central even number $4n-2$ of unit partial quotients in the symmetric part, then $R = p^2 + q^2$, where $p > 2q > 0$ and the complete quotient $\frac{q+\sqrt{R}}{p}$ occurs just once in the recurring period.

Proof. Let $\frac{\sqrt{R}}{Q_0} = (a_0, \underset{*}{a_1}, \ldots, a_{k-1}, \underset{*}{2a_0})$. (1)

From Lemma 2, a complete quotient, say ξ of the form in question in (1) cannot have $2a_0$ (obviously $\neq 1$) as its first partial quotient, so that we may write $\xi = (\underset{*}{a_v}, \ldots, \underset{*}{a_{v-1}})$, where $a_v \neq 2a_0$. From the equality of the first and last Q's in ξ, we must have $Q_v = Q_{v-1}$ in (1), which implies, by a well-known theorem of Muir * that the period-length k is odd and $v = \frac{k+1}{2}$; and in this case, it is easily seen that $\xi = \frac{P_v + \sqrt{R}}{Q_v}$ and $R = P_v^2 + Q_v^2$.

Further, there cannot be another complete quotient of the same form in the recurring period, since it is possible only when the number of terms in the recurring period is even.

We infer therefore that $\xi_{\frac{k+1}{2}} = (\underset{*}{a}_{\frac{k+1}{2}}, \ldots, a_{k-1}, 2a_0, a_1, \ldots, \underset{*}{a}_{\frac{k+1}{2}})$, where an odd number $2n-1$ of unit partial quotients must begin with $a_{\frac{k+1}{2}}$ and also an equal odd number $2n-1$ of such partial quotients end with $a_{\frac{k+1}{2}}$.

Thus $\frac{\sqrt{R}}{Q_0}$ must contain in its period, an even number $4n-2$ of unit partial quotients in the centre of the symmetric portion, as, for example,

(i) $\sqrt{58} = (7, \underset{*}{1}, 1, 1, 1, 1, 1, \underset{*}{14})$; (ii) $\sqrt{97} = (9, \underset{*}{1}, 5, 1, 1, 1, 1, 1, 1, 5, 1, \underset{*}{18})$.

In this case, $\xi_{\frac{k+1}{2}}$ is of the form $\frac{p+\sqrt{p^2+q^2}}{q}$, where $v = \frac{k+1}{2}, p = P_v, q = Q_v$ and $\xi_{\frac{k+1}{2}} < (1_\infty)$, as the continued fraction begins with an odd number of unit partial quotients.

Hence $\frac{q+\sqrt{p^2+q^2}}{p} < \frac{1+\sqrt{5}}{2}, \frac{-q+\sqrt{p^2+q^2}}{p} > \frac{-1+\sqrt{5}}{2}$, so that subtracting the second from the first, gives $2q/p < 1$ and obviously p and q are positive in a recurring period.

This completes our proof.

5.5.1. We will now point out an application of the last two lemmas to the most rapidly convergent continued fractions. Tietze* has shown that such continued fractions are characterised by the property that the complete quotients are, after a point, always greater than $\frac{1+\sqrt{5}}{2}$. The B.c.f.'s are therefore of this class. We have proved elsewhere† that the only transformations (apart from the P-transformation) which convert a simple continued fraction into one of the most rapidly convergent h.r.c.f.'s are the annihilatory transformations, which we have called the C_1, C_2 and C_1C_2 types. The effect of an annihilatory transformation applied to a unit partial quotient is obviously to increase the following complete quotient by 1, without affecting the preceding complete quotient.

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*Vide p.91, Perron, *loc. cit.*

*Tietze, H., *Monatshefte für Mathematik und Physik*, 1913, 24

†Ayyangar, A.A.K., *Maths. Student*, 1938, 6

From these considerations, we see that a complete quotient of the form $\frac{q+p+\sqrt{p^2+q^2}}{p}$ will occur in any most rapidly convergent h.r.c.f. development (not involving a P-transformation) of $\frac{\sqrt{R}}{Q_0}$ (>1 and in standard form), when and only when either $\frac{q+\sqrt{p^2+q^2}}{p}$ or $\frac{q+p+\sqrt{p^2+q^2}}{p}$ occurs in the simple continued fraction development. But $\frac{q+p+\sqrt{p^2+q^2}}{p}$ is not a reduced surd in Perron's sense‡ and therefore cannot occur in the recurring period of the simple continued fraction, while $\frac{q+\sqrt{p^2+q^2}}{p}$ will occur just once in the recurring period under the conditions of Lemma 3.

Hence, every most rapidly convergent h.r.c.f. development of $\frac{\sqrt{R}}{Q_0}$ (not involving a P-transformation) will contain in its period $\frac{q+p+\sqrt{p^2+q^2}}{p}$ as a complete quotient just once when the unit partial quotient corresponding to $\frac{q+\sqrt{p^2+q^2}}{p}$ in the simple continued fraction is not annihilated.

If $\frac{\sqrt{R}}{Q_0} = (a_0, a_1, \ldots, a_p, 1_{[4l+2]}, a_p, \ldots, a_1, 2a_0)$, where ξ_{p+2l+2} is the only complete quotient of the form $\frac{q+\sqrt{p^2+q^2}}{p}$, the result of applying the C_1-transformation gives the complete quotient $1+\xi_{p+2l+2}$, while the C_2-transformation will annihilate the unit partial quotient corresponding to ξ_{p+2l+2} and so there will be no complete quotient of the form in question.

To preserve the complete quotient, we may also apply the eclectic transformation C_1C_2, provided that the C_1 process is continued at least until it annihilates the $(2l+1)$-th central unit quotient Hence we may state that it is possible to have a complete quotient of the form in question in the B.c.f. development as well as in the continued fraction to the nearest integer, but not in the singular continued fraction (all of which do not involve the P-transformation*). (Inserted by Keith Matthews: a less vague, but related explanation can be based on §40 of Perron's *Kettenbrüchen* (1954) Band 1, 147-154 using Perron's T_1 and T_2 transformations. See the appendix.)

5.6. We are now in a position to resume our original thread of discussion and study the nature of the recurring period of the B.c.f. development of $\frac{\sqrt{R}}{Q_0}$. We recognize three possible types:

‡ *Vide* p.79, Perron, *loc. cit.*

* *Vide Maths. Student*, **6**, 63 and *J. Mysore Univ.* Vol. 1 Part II, Note (2), Th. II

Type I. This occurs when the recurring cycle does not contain any complete quotient of the form $(1-g)^{-1}$, i.e., $\frac{q+p+\sqrt{p^2+q^2}}{p}$, where g is a special critical fraction pertaining to R. Evidently this type must occur when R cannot be expressed as the sum of two squares, or when $\frac{\sqrt{R}}{Q_0}$ does not satisfy the conditions of Lemma 3. We will presently show that the characteristic property of this type is that it simulates the simple continued fraction period in its symmetries and also in the property of the last partial quotient. Thus

(i) $b_{v-1} = b_{k-v}, \quad Q_{v-1} = Q_{k-v}$ $(v = 2, 3, \ldots, k-1)$;

(ii) $\epsilon_v = \epsilon_{k-v}, \quad P_v = P_{k-v}$ $(v = 1, 2, \ldots, k-1)$;

e.g. $\sqrt{46} = 7 - \frac{1|}{\underset{*}{|5}} - \frac{1|}{|2} + \frac{1|}{|2} + \frac{1|}{|6} + \frac{1|}{|2} + \frac{1|}{|2} - \frac{1|}{|5} - \frac{1|}{\underset{*}{|14}}$.

Type II. This occurs when the recurring cycle contains a complete quotient of the form $\xi = \frac{q+p+\sqrt{p^2+q^2}}{p}$, where $p > 2q > 0$. We call this *almost symmetrical*, as the symmetries are slightly disturbed. It has the form

$$\frac{\sqrt{R}}{Q_0} = b_0 + \frac{\epsilon_1|}{\underset{*}{|b_1}} + \cdots + \frac{\epsilon_{\frac{k-3}{2}}|}{|b_{\frac{k-3}{2}}} - \frac{1|}{|2} + \frac{1|}{|b_{\frac{k+1}{2}}} + \frac{\epsilon_{\frac{k+3}{2}}|}{|b_{\frac{k+3}{2}}} + \cdots + \frac{\epsilon_{k-1}|}{\underset{*}{|2b_0}},$$

where the period $k-1$ is even and $b_{\frac{k-3}{2}} = b_{\frac{k+1}{2}} + 1$, $P_{\frac{k-1}{2}} \neq P_{\frac{k+1}{2}}$, but otherwise has the symmetries of Type I. Also $\frac{P_{\frac{k-1}{2}} + \sqrt{R}}{Q_{\frac{k-1}{2}}} = \xi$. e.g. $\sqrt{58} = 8 - \frac{1|}{\underset{*}{|3}} - \frac{1|}{|2} + \frac{1|}{|2} - \frac{1|}{\underset{*}{|16}}$.[5]

Type III. This has only two terms in the recurring period and has the form $\sqrt{n^2+n+\frac{1}{2}} = n + 1 - \frac{1|}{\underset{*}{|2}} + \frac{1|}{\underset{*}{|2n+1}}$.

5.6.1. [6] We need a result analogous to Satz 6, p.83, Perron.

Suppose

$$\xi_0 = b_0 + \frac{\epsilon_1}{\xi_1}, \xi_1 = b_1 + \frac{\epsilon_2}{\xi_2}, \ldots, \xi_{k-1} = b_{k-1} + \frac{\epsilon_k}{\xi_k},$$

[5] Selenius noticed that the + was a − in the original

[6] Comment inserted by Keith Matthews

is a periodic B.c.f. expansion, where $\xi_k = \xi_0$.

Let $\zeta_v = -\epsilon_{k-v}/\overline{\xi}_{k-v}, v = 0, 1, \ldots, k$, where $\epsilon_0 = \epsilon_k$. Then $\zeta_{v-1} = b_{k-v} + \frac{\epsilon_{k-v}}{\zeta_v}$ and

$$\frac{-\epsilon_k}{\overline{\xi_0}} = \zeta_0 = b_{k-1} + \frac{\epsilon_{k-1}}{\zeta_1}, \zeta_1 = b_{k-2} + \frac{\epsilon_{k-2}}{\zeta_2}, \ldots, \zeta_{k-1} = b_0 + \frac{\epsilon_k}{\zeta_k}.$$

Also $\zeta_k = \zeta_0$.

Then if none of $\xi_0, \xi_1, \ldots, \xi_{k-1}$ has the form $(1-g)^{-1}$, where g is a special critical fraction, then by Theorem VIII, the above recurrences also form a B.c.f. expansion.

Let $\xi_0 = \frac{\sqrt{R}}{Q_0} = b_0 + \frac{\epsilon_1|}{|\underset{*}{b_1}} + \cdots \frac{\epsilon_{k-1}|}{|\underset{*}{b_{k-1}}}$, $\xi_v = \frac{P_v + \sqrt{R}}{Q_v}$ and $\zeta_v = -\frac{\epsilon_{k-v}}{\overline{\xi}_{k-v}}, (v = 0, 1, \ldots, k-1)$, where ξ_v is the v-th successor of ξ_0 and $\xi_k = \xi_1$.

Then, as in the simple continued fraction, (see above discussion) we have

$$\zeta_{v-1} = b_{k-v} + \frac{\epsilon_{k-v}}{\zeta_v} = \frac{P_{k-v+1} + \sqrt{R}}{Q_{k-v}}$$
$$\zeta_0 = b_{k-1} + \frac{\epsilon_{k-1}|}{|b_{k-2}} + \cdots \frac{\epsilon_1|}{|\zeta_{k-1}}.$$

But $\zeta_{k-1} = \frac{-\epsilon_1}{\overline{\xi}_1} = \frac{-\epsilon_k}{\overline{\xi}_k} = \zeta_0$. Hence

$$\zeta_0 = \underset{*}{b_{k-1}} + \frac{\epsilon_{k-1}|}{|b_{k-2}} + \cdots \frac{\epsilon_2|}{|\underset{*}{b_1}} + \frac{\epsilon_1|}{\zeta_0}. \qquad (1)$$

By Theorem VIII, ζ_v is the Bhaskara successor of ζ_{v-1} in all cases, except when $Q^2_{k-v-1} + \frac{1}{4}Q^2_{k-v-2} = R$, which implies that $\epsilon_{k-v-1} = -1, \epsilon_{k-v} = 1$ and ξ_{k-v-1} is of the form $(1-g)^{-1}$, g being a special critical fraction.

When no successor (immediate or remote) of $\sqrt{R}/Q = \sqrt{D}$ (say) ,

$$\sqrt{D} = b_0 + \frac{\epsilon_1|}{|\underset{*}{b_1}} + \cdots \frac{\epsilon_{k-1}|}{|\underset{*}{b_{k-1}}}, \qquad (2)$$

is of the form in question, we may write

$$\frac{\epsilon_1}{\sqrt{D} - b_0} = \underset{*}{b_1} + \frac{\epsilon_2|}{|b_2} + \cdots \frac{\epsilon_{k-1}|}{|\underset{*}{b_{k-1}}}.$$

But $\zeta_0 = \frac{-\epsilon_k}{\bar{\xi}_k} = \frac{-\epsilon_k}{\bar{\xi}_1} = \epsilon_k\epsilon_1(\sqrt{D} + b_0)$.

Hence (1) gives

$$\epsilon_k\epsilon_1(\sqrt{D} + b_0) = \underset{*}{b_{k-1}} + \frac{\epsilon_{k-1}|}{|b_{k-2}} + \cdots \frac{\epsilon_2|}{|\underset{*}{b_1}}. \qquad (3)$$

Since the r.h.s. is positive, $\epsilon_k\epsilon_1 = 1$.

Comparing (2) and (3), which are both B.c.f.'s, we get $b_{k-1} = 2b_0$ and the following symmetries:

$$\begin{array}{lcll} b_{v-1} & = & b_{k-v} & (v = 2, 3, \ldots k-1); \\ Q_{v-1} & = & Q_{k-v} & (v = 2, 3, \ldots k-1); \\ \epsilon_v & = & \epsilon_{k-v} & (v = 1, 2, \ldots k-1); \\ P_v & = & P_{k-v} & (v = 1, 2, \ldots k-1). \end{array}$$

When k is even, or the number $k-1$ of terms in the recurring period is odd, two consecutive b's and two consecutive Q's are equal, *viz.*, $b_{\frac{k-2}{2}} = b_{\frac{k}{2}}$, $Q_{\frac{k-2}{2}} = Q_{\frac{k}{2}}$.

When k is odd, or the number $k-1$ of terms in the recurring period is even, two consecutive a's and two consecutive P's are equal, *viz.*, $\epsilon_{\frac{k-1}{2}} = \epsilon_{\frac{k+1}{2}}$, $P_{\frac{k-1}{2}} = P_{\frac{k+1}{2}}$.

Conversely, if two consecutive Q's are equal in the recurring cycle, say $Q_v = Q_{v-1}$, then $\xi_v = \frac{P_{k-v}+\sqrt{R}}{Q_{k-v}} = \frac{P_v+\sqrt{R}}{Q_v} = \xi_v$, so that $v = k/2$ and k is even. Similarly for two consecutive P's, $v = \frac{k+1}{2}$ and k is odd.

Theorem XIV. If $\frac{\sqrt{R}}{Q_0}(> 1)$ develops as Type I B.c.f. and the number of terms in the recurring cycle is odd, then R is either a sum of two squares or a composite number of 3.

Proof. Let k' be the cycle-length, k' odd. Then with $v = \frac{k'-1}{2}$,

$$P_{v+1}^2 + \epsilon_{v+1}Q_vQ_{v+1} = R \text{ and } Q_v = Q_{v+1}.$$

If $\epsilon_{v+1} = 1$, $R = P_{v+1}^2 + Q_{v+1}^2$.
If $\epsilon_{v+1} = -1$, $P_{v+1}^2 - Q_{v+1}^2 = R$. So if R is a prime, we have

$$P_{v+1} + Q_{v+1} = R \text{ and } P_{v+1} - Q_{v+1} = 1,$$

so that $Q_{v+1} = \frac{R-1}{2} < \sqrt{R}$ and therefore $R = 3$ or 5. In both these cases, it is easily verifed that $k' = 1$.

When R is neither 3 nor a sum of two squares, then $\epsilon_{v+1} = -1$ and R is composite.

Corollary. When R is a prime other than 3 and is not the sum of two squares, then the cycle-length k' is even.

5.6.2. If in the B.c.f. development of $\sqrt{R}/Q_0$ given in (2) of 5.6.1, ξ_{k-2} happens to be of the form $(1-g)^{-1} = \frac{p+q+\sqrt{R}}{p}, p > 2q > 0$, then $\xi_{k-1} = \overline{-g} = \frac{p-q+\sqrt{R}}{2q}$ is the conjugate of $-g$ (*vide* Theorem X) and being the predecessor of ξ_k, is also of the form $\sqrt{D} + \mu$, where μ is an integer; i.e., $\frac{p-q+\sqrt{R}}{2q} = \mu + \sqrt{D}$. Hence $p - q = 2qn$, n an integer and $p > 2q > 0$. Hence $p = (2n+1)q, R = p^2 + q^2 = q^2(4n^2 + 4n + 2)$; also $\gcd(Q_0, R/Q_0) = 1$, so $q = 1$ and $p = 2n + 1$. $\xi_{k-1} = n + \frac{\sqrt{R}}{2q}$, so that $\sqrt{D}$ is of the form $\sqrt{4n^2 + 4n + 2}/2$.

The Bcf development of $\sqrt{4n^2 + 4n + 2}/2$ is $n + 1 - \underset{*}{\frac{1|}{|2}} + \underset{*}{\frac{1|}{|2n+1}}$, with $\xi_1 = \frac{2n+2+\sqrt{R}}{2n+1}, \xi_2 = \frac{2n+\sqrt{R}}{2}$ and $\xi_3 = \xi_1$.

This is what we have called Type III.

5.6.3. As we have already seen, the recurring period in Type II will contain one and only one complete quotient of the form ${\xi'}_0 = \frac{p+q+\sqrt{R}}{p}, p > 2q > 0$ and therefore, the recurring cycle will be merely a cyclic permutation of that of this complete quotient.

By Theorem XIII, ${\xi'}_0$ develops as a pure recurring B.c.f. with period k', We will now proceed to study its nature, on the assumption that ${\xi'}_v$ is only of the form $(1-g)^{-1}$ when $v = tk', t \geq 0$.

As observed in the proof of Theorem X,

$$\begin{aligned} \xi'_0 &= 2 + \frac{1}{\overline{-g}} = 2 + \frac{2q}{p - q + \sqrt{R}} \\ &= \underset{*}{2} + \frac{1|}{|b'_1} + \frac{\epsilon'_2|}{|b'_2} + \cdots + \underset{*}{\frac{\epsilon'_{k'-1}|}{|b'_{k'-1}}}. \end{aligned}$$

Hence

$$\overline{-g} = b'_1 + \frac{\epsilon'_2|}{|b'_2} + \cdots + \frac{\epsilon'_{k'-2}|}{|b'_{k'-2}} + \frac{\epsilon'_{k'-1}|}{|\xi_{k'-1}} \qquad (1)$$

Now by Theorem VIII, the following are Bhaskara expansions, since ξ'_v is not of the form $(1-g)^{-1}$ for $v = 1, 2, \ldots, k'-1$:

$$\zeta'_0 = b'_{k'-1} + \frac{\epsilon'_{k'-1}}{\zeta'_1}, \cdots, \zeta'_{k'-3} = b'_2 + \frac{\epsilon'_2}{\zeta'_{k'-2}}.$$

or

$$\zeta'_0 = b'_{k'-1} + \frac{\epsilon'_{k'-1}|}{|b'_{k'-2}} + \cdots + \frac{\epsilon'_3|}{|b'_2} + \frac{\epsilon'_2|}{|\zeta'_{k'-2}} \qquad (2)$$

But $\zeta'_0 = \overline{-g} + 1$. For, noting that $P'_{k'} = P'_0 = p + q$ and $Q'_{k'} = Q'_0 = p$,

$$\begin{aligned} p^2 + q^2 = R &= {P'_{k'}}^2 + \epsilon'_{k'} Q'_{k'} Q'_{k'-1} \\ &= (p+q)^2 + \epsilon'_{k'} p Q'_{k'-1} \\ -2pq &= \epsilon'_{k'} p Q'_{k'-1} \\ -2q &= \epsilon'_{k'} Q'_{k'-1}. \end{aligned}$$

Hence $\epsilon'_{k'} = -1$ and $Q'_{k'-1} = 2q$.

Hence

$$\zeta'_0 = \frac{P'_{k'} + \sqrt{R}}{Q'_{k'-1}} = \frac{p + q + \sqrt{R}}{2q} = \frac{p - q + \sqrt{R}}{2q} + 1 = \overline{-g} + 1.$$

Then (2) gives

$$\overline{-g} = -1 + b'_{k'-1} + \frac{\epsilon'_{k'-1}|}{|b'_{k'-2}} + \cdots + \frac{\epsilon'_3|}{|b'_2} + \frac{\epsilon'_2|}{|\zeta'_{k'-2}} \qquad (3)$$

We can equate the first $k'-1$ complete quotients and the first $k'-2$ terms of (1) and (3), to obtain the following properties of B.c.f of $(1-g)^{-1}$:

(i) $-1 + b'_{k'-1} = b'_1$;

(ii) the following symmetries hold if $k' > 2$:

$$\begin{array}{rcll} b'_v &=& b'_{k'-v} & (v = 2, 3, \ldots k'-2); \\ Q'_v &=& Q'_{k'-v} & (v = 1, 2, \ldots k'-1); \\ \epsilon'_v &=& \epsilon'_{k'-v+1} & (v = 2, 3, \ldots k'-1); \\ P'_v &=& P'_{k'-v+1} & (v = 2, 3, \ldots k'-1). \end{array}$$

(iii) Also

$$P'_1 = p - q, Q'_1 = 2q, P'_{k'-1} = q(2n-1) - p, Q'_{k'-1} = 2q,$$

where by the proof of Theorem XIII, $n = b'_{k'-1}$ is the integer just greater than p/q when p is not divisible by q and $n = p/q$ otherwise.

Thus if $p > 2q > 0$, period-length k' and if ξ_v is only of the form $(1-g)^{-1}$ when $v = tk', t \geq 0$, we have

$$\frac{p+q+\sqrt{p^2+q^2}}{p} = \underset{*}{2} + \frac{1|}{|b'_1} + \frac{\epsilon'_2|}{|b'_2} + \cdots + \frac{\epsilon'_{k'-2}|}{|b'_{k'-2}} + \frac{\epsilon'_{k'-1}|}{\underset{*}{|b'_1 + 1}} - \frac{1|}{|2} + \frac{1|}{|b'_1} + \cdots$$

As in §5.6.1, we can prove that two consecutive Q's will be equal, only when k' is odd and that two consecutive P's will be equal, only when k' is even. For example, if $P'_v = P'_{v+1}$, then

$$\xi'_v = \frac{P'_v + \sqrt{R}}{Q'_v} = \frac{P'_{v+1} + \sqrt{R}}{Q'_v} = \frac{P'_{k'-v} + \sqrt{R}}{Q'_{k'-v}},$$

so that $v = k' - v$, or $v = k'/2$, i.e., k' is even. Conversely if k' is even, $P'_{\frac{k'}{2}} = P'_{\frac{k'+2}{2}}$.

Similarly, if $Q'_v = Q'_{v-1}$, then $\xi'_v = \xi'_{k'-v+1}$ and $v = (k'+1)/2$, i.e., k' is odd. Conversely if k' is odd, $Q'_{\frac{k'+1}{2}} = Q'_{\frac{k'-1}{2}}$. [7]

5.6.4. Reverting to the B.c.f. development of $\sqrt{D}(= \sqrt{R}/Q_0) = \xi_0$ and following the notation of §5.6.1, we assume that $\xi_{k-v-1}, 0 \leq v \leq k-2$ is the only complete quotient of the form $\frac{p+q+\sqrt{p^2+q^2}}{p}, p > 2q > 0$, in the period of $\sqrt{D}$.

Now $v = 0$ implies $\xi_{k-1} = (1-g)^{-1}$, $\xi_1 = \overline{-g}$. But this would imply $\overline{-g}$ was a successor of $\sqrt{D}$, contradicting Theorem X, Corollary 2.

Also $v = 1$ leads to a Type III expansion.

Hence $1 < v \leq k-2$ and $(1-g)^{-1} = \xi_{k-v-1} = \xi_t, 1 \leq t < k-2$. Then

$$b_{k-1} = 2b_0, \epsilon_1 = \epsilon_{k-1}, P_1 = P_{k-1}, Q_1 = Q_{k-2}.$$

[7]Examples (KRM): $\frac{19+\sqrt{221}}{14}, p = 14, q = 5, k' = 3$; $\frac{13+\sqrt{97}}{9}, p = 9, q = 4, k' = 6$.

(AAK states in addition, that $b_{k-2} = b_1$, but this would hold only if the period-length $k-1 \geq 6$.)

For ξ_{k-2} is not of the form $\frac{p+q+\sqrt{p^2+q^2}}{p}$, $p > 2q > 0$ and hence by Theorem VIII, we have the Bhaskara expansion

$$\frac{P_k + \sqrt{R}}{Q_{k-1}} = b_{k-1} + \frac{\epsilon_{k-1}}{\frac{P_{k-1}+\sqrt{R}}{Q_{k-2}}} \qquad (1)$$

Now $P_k = P_1$. Also $P_0 = 0$ and $P_1 + P_0 = b_0 Q_0$. Hence $P_1 = b_0 Q_0$.

Also $R = P_1^2 + \epsilon_1 Q_0 Q_1 = P_k^2 + \epsilon_k Q_{k-1} Q_k$. Hence, as $\epsilon_1 = \epsilon_k$ and $Q_1 = Q_k$, we have $Q_{k-1} = Q_0$. Hence

$$\frac{P_k + \sqrt{R}}{Q_{k-1}} = \frac{b_0 Q_0 + \sqrt{R}}{Q_0} = b_0 + \frac{\sqrt{R}}{Q_0}. \qquad (2)$$

Then (1) and (2) give

$$b_{k-1} + \frac{\epsilon_{k-1}}{\frac{P_{k-1}+\sqrt{R}}{Q_{k-2}}} = 2b_0 + \frac{\epsilon_1}{\frac{P_1+\sqrt{R}}{Q_1}}. \qquad (3)$$

Hence

$$b_{k-1} = 2b_0, \epsilon_{k-1} = \epsilon_1, \frac{P_{k-1} + \sqrt{R}}{Q_{k-2}} = \frac{P_1 + \sqrt{R}}{Q_1}.$$

Consequently $P_{k-1} = P_1$ and $Q_{k-2} = Q_1$.

Now the sequence

$$\xi_1, \ldots, \xi_{k-1} = (1-g)^{-1} \qquad (\alpha)$$

of complete quotients for $\sqrt{D}$ is obtained from the sequence

$$\xi'_1 = \overline{-g}, \ldots, \xi'_{k-1} = (1-g)^{-1} \qquad (\beta)$$

of complete quotients for $(1-g)^{-1}$, by cyclic permutation.

Hence

$$\xi_{t+1} = \xi'_1, \xi_{t+2} = \xi'_2, \ldots, \xi_{k-1} = \xi'_{k-1-t}, \xi_1 = \xi_k = \xi'_{k-t}.$$

But $P_{k-1} = P_1$, so $P'_{k-1-t} = P'_{k-t}$. Consequently the period $k-1 = k'$ is even and $k' - t = k'/2$. Hence $t = k'/2$. (See Figures 2 and 3 below where $k' = 6$.)

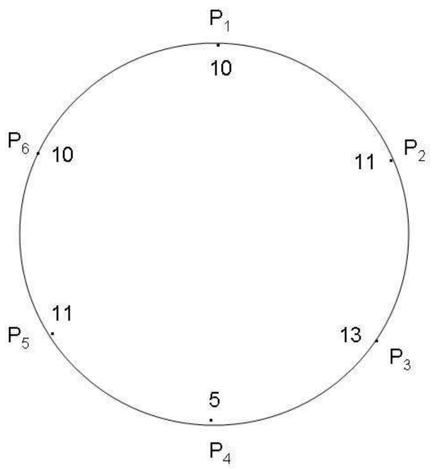

Figure 2: P's for complete quotients $\xi_v = \frac{P_v+\sqrt{97}}{Q_v}$ of $\xi_0 = \sqrt{97}$

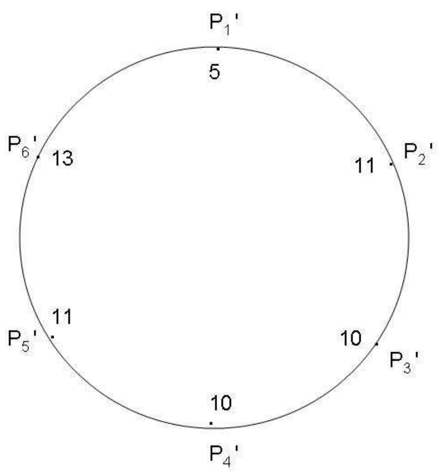

Figure 3: P''s for complete quotients $\xi'_v = \frac{P'_v+\sqrt{97}}{Q'_v}$ of $\xi'_0 = \frac{13+\sqrt{97}}{9} = \xi_3$

Then we have a periodic expansion for Type II of the form

$$\sqrt{D} = b_0 + \frac{\epsilon_1|}{|\underset{*}{b_1}} + \cdots + \frac{\epsilon_{\frac{k-3}{2}}|}{|b_{\frac{k-3}{2}}} - \frac{1|}{|2} + \frac{1|}{|b_{\frac{k+1}{2}}} + \cdots + \frac{\epsilon_{k-1}|}{|\underset{*}{2b_0}} + \frac{\epsilon_1|}{|b_1} + \cdots$$

having an even number of recurring terms and possessing the same symmetries as Type I, apart from the following exceptions:

$$b_{\frac{k-1}{2}} = 2, \epsilon_{\frac{k-1}{2}} = -1, \epsilon_{\frac{k+1}{2}} = 1, b_{\frac{k-3}{2}} = b_{\frac{k+1}{2}} + 1, P_{\frac{k-1}{2}} \neq P_{\frac{k+1}{2}},$$

which justify our characterisation of this type as *almost* symmetric.

Example[8]. $R = 97$, $Q_0 = 1$.

$$\sqrt{97} = 10 - \frac{1|}{|\underset{*}{7}} \frac{-1|}{|3} \frac{-1|}{|2} \frac{+1|}{|2} \frac{-1|}{|7} \frac{-1|}{|\underset{*}{20}}$$

$$\frac{13+\sqrt{97}}{9} = 2 + \frac{1|}{|\underset{*}{2}} \frac{-1|}{|7} \frac{-1|}{|20} \frac{-1|}{|7} \frac{-1|}{|3} \frac{-1|}{|\underset{*}{2}}.$$

It may be useful to telescope the results of this section applicable to the case of $\sqrt{R}$, where R is a non-square positive integer, in the form of a theorem.

Theorem XV. The period of the B.c.f. development of $\sqrt{R}$ is either a completely symmetrical type simulating the corresponding simple continued fraction, or an almost symmetrical type, consisiting of an even numbe rof partial quotients, say, 2ν, with a central sequence of three unsymmetrical terms of the form $\frac{\epsilon_{\nu-1}|}{|b_{\nu-1}} \frac{-1|}{|2} \frac{+1|}{|b_{\nu-1}-1}$.

Corollary. In the almost symmetrical type B.c.f. expansion of $\sqrt{R}$ with period consisting of $2v$ terms, $Q_v > 4$.

Proof. For $\frac{P_v+\sqrt{R}}{Q_v} = \frac{p+q+\sqrt{p^2+q^2}}{p}$ and $Q_v = p > 2q$.

If $q = 1$, then $\sqrt{R} = \sqrt{p^2+1} = p + \frac{1}{\underset{*}{2p}}$, which is not of type II. Hence $q \geq 2$ and $Q_v > 4$. In fact when $Q_v = 5 = p$, we must have $q = 2$ and $R = 29$. $\sqrt{29} = 5 + \frac{1|}{|\underset{*}{3}} \frac{-1|}{|2} \frac{+1|}{|2} \frac{+1|}{|\underset{*}{10}}$.

[8] Keith Matthews

We give below a table of the B.c.f.'s for the square-roots of non-square integers less than 100.

R	B.c.f.	R	B.c.f.
2	$1+1/\underset{*}{2}$	23	$5-1/\underset{*}{5}-1/\underset{*}{10}$
3	$2-1/\underset{*}{4}$	24	$5-1/\underset{*}{10}$
5	$2+1/\underset{*}{4}$	26	$5+1/\underset{*}{10}$
6	$2+1/\underset{*}{2}+1/\underset{*}{4}$	27	$5+1/\underset{*}{5}+1/\underset{*}{10}$
7	$3-1/\underset{*}{3}-1/\underset{*}{6}$	28	$5+1/\underset{*}{3}+1/\underset{*}{2}+1/3+1/\underset{*}{10}$
8	$3-1/\underset{*}{6}$	29	$5+1/\underset{*}{3}-1/2+1/2+1/\underset{*}{10}$
10	$3+1/\underset{*}{6}$	30	$5+1/\underset{*}{2}+1/\underset{*}{10}$
11	$3+1/\underset{*}{3}+1/\underset{*}{6}$	31	$6-1/\underset{*}{2}+1/3+1/5+1/3+1/2-1/\underset{*}{12}$
12	$3+1/\underset{*}{2}+1/\underset{*}{6}$	32	$6-1/\underset{*}{3}-1/\underset{*}{12}$
13	$4-1/\underset{*}{2}+1/2-1/\underset{*}{8}$	33	$6-1/\underset{*}{4}-1/\underset{*}{12}$
14	$4-1/\underset{*}{4}-1/\underset{*}{8}$	34	$6-1/\underset{*}{6}-1/\underset{*}{12}$
15	$4-1/\underset{*}{8}$	35	$6-1/\underset{*}{12}$
17	$4+1/\underset{*}{8}$	37	$6+1/\underset{*}{12}$
18	$4+1/\underset{*}{4}+1/\underset{*}{8}$	38	$6+1/\underset{*}{6}+1/\underset{*}{12}$
19	$4+1/\underset{*}{3}-1/5-1/3+1/\underset{*}{8}$	39	$6+1/\underset{*}{4}+1/\underset{*}{12}$
20	$4+1/\underset{*}{2}+1/\underset{*}{8}$	40	$6+1/\underset{*}{3}+1/\underset{*}{12}$
21	$5-1/\underset{*}{2}+1/2+1/2-1/\underset{*}{10}$	41	$6+1/\underset{*}{2}+1/2+1/\underset{*}{12}$
22	$5-1/\underset{*}{3}+1/4+1/3-1/\underset{*}{10}$	42	$6+1/\underset{*}{2}+1/\underset{*}{12}$

R	B.c.f.	R	B.c.f.
43	$7-1/\underset{*}{2}+1/4-1/7-1/4+1/2-1/\underset{*}{14}$	72	$8+1/\underset{*}{2}+1/\underset{*}{16}$
44	$7-1/\underset{*}{3}-1/4-1/3-1/\underset{*}{14}$	73	$9-1/\underset{*}{2}+1/5+1/5+1/2-1/\underset{*}{18}$
45	$7-1/\underset{*}{3}+1/2+1/3-1/\underset{*}{14}$	74	$9-1/\underset{*}{2}+1/2-1/\underset{*}{18}$
46	$7-1/\underset{*}{5}-1/2+1/2+1/6+1/2+1/2-1/5-1/\underset{*}{14}$	75	$9-1/\underset{*}{3}-1/\underset{*}{18}$
47	$7-1/\underset{*}{7}-1/\underset{*}{14}$	76	$9-1/\underset{*}{3}+1/2-1/6+1/4+1/6-1/2+1/3-1/\underset{*}{18}$
48	$7-1/\underset{*}{14}$	77	$9-1/\underset{*}{4}+1/2+1/4-1/\underset{*}{18}$
50	$7+1/\underset{*}{14}$	78	$9-1/\underset{*}{6}-1/\underset{*}{18}$
51	$7+1/\underset{*}{7}+1/\underset{*}{14}$	79	$9-1/\underset{*}{9}-1/\underset{*}{18}$
52	$7+1/\underset{*}{5}-1/4-1/5+1/\underset{*}{14}$	80	$9-1/\underset{*}{18}$
53	$7+1/\underset{*}{4}-1/2+1/3+1/\underset{*}{14}$	82	$9+1/\underset{*}{18}$
54	$7+1/\underset{*}{3}-1/8-1/3+1/\underset{*}{14}$	83	$9+1/\underset{*}{9}+1/\underset{*}{18}$
55	$7+1/\underset{*}{2}+1/2+1/2+1/\underset{*}{14}$	84	$9+1/\underset{*}{6}+1/\underset{*}{18}$
56	$7+1/\underset{*}{2}+1/\underset{*}{14}$	85	$9+1/\underset{*}{5}-1/2+1/4+1/\underset{*}{18}$
57	$8-1/\underset{*}{2}+1/4+1/2-1/\underset{*}{16}$	86	$9+1/\underset{*}{4}-1/3-1/10-1/3-1/4+1/\underset{*}{18}$
58	$8-1/\underset{*}{3}-1/2+1/2-1/\underset{*}{16}$	87	$9+1/\underset{*}{3}+1/\underset{*}{18}$
59	$8-1/\underset{*}{3}+1/7+1/3-1/\underset{*}{16}$	88	$9+1/\underset{*}{3}-1/3-1/3+1/\underset{*}{18}$
60	$8-1/\underset{*}{4}-1/\underset{*}{16}$	89	$9+1/\underset{*}{2}+1/3+1/3+1/2+1/\underset{*}{18}$
61	$8-1/\underset{*}{5}+1/4-1/3+1/3-1/4+1/5-1/\underset{*}{16}$	90	$9+1/\underset{*}{2}+1/\underset{*}{18}$
62	$8-1/\underset{*}{8}-1/\underset{*}{16}$	91	$10-1/\underset{*}{2}+1/6-1/6+1/2-1/\underset{*}{20}$
63	$8-1/\underset{*}{16}$	92	$10-1/\underset{*}{2}+1/2+1/4+1/2+1/2-1/\underset{*}{20}$
65	$8+1/\underset{*}{16}$	93	$10-1/\underset{*}{3}-1/5+1/6+1/5-1/3-1/\underset{*}{20}$
66	$8+1/\underset{*}{8}+1/\underset{*}{16}$	94	$10-1/\underset{*}{3}+1/3+1/2-1/7-1/10-1/7-1/2+1/3+1/3-1/\underset{*}{20}$
67	$8+1/\underset{*}{5}+1/2+1/2-1/9-1/2+1/2+1/5+1/\underset{*}{16}$	95	$10-1/\underset{*}{4}-1/\underset{*}{20}$
68	$8+1/\underset{*}{4}+1/\underset{*}{16}$	96	$10-1/\underset{*}{5}-1/\underset{*}{20}$
69	$8+1/\underset{*}{3}+1/4-1/6-1/4+1/3+1/\underset{*}{16}$	97	$10-1/\underset{*}{7}-1/3-1/2+1/2-1/7-1/\underset{*}{20}$
70	$8+1/\underset{*}{3}-1/4-1/3+1/\underset{*}{16}$	98	$10-1/\underset{*}{10}-1/\underset{*}{20}$
71	$8+1/\underset{*}{2}+1/3-1/9-1/3+1/2+1/\underset{*}{16}$	99	$10-1/\underset{*}{20}$

(See footnote[9])

[9]For $D = 86$, Selenius noticed that the second $+$ was a $-$ in the original

The basic elements of the theory are now fairly complete, and it should be obvious that the B.c.f. has a complicated indviduality of its own, that claims recognition and cannot easily be brushed aside by such remarks a "Bhaskara's method is the same as that rediscovered by Lagrange". We have only constructed "an arch, wherethro' gleam untravelled and partly travelled regions", such as the character of the acyclic part, the transformations that convert the simple continued fraction into the continued fraction to the nearest square, and associated quadratic forms. These difficult problems need further investigation.

Appendix (supplement to 5.5.1)

We consider the B.c.f. expansion of $\xi_0 = \frac{p+q+\sqrt{p^2+q^2}}{p}$, where $p > 2q > 0$ and consider the conversion of ξ_0 to a simple continued fraction.

Suppose $\xi_v = \frac{p'+q'+\sqrt{p'^2+q'^2}}{p'}, p' > 2q' > 0$ occurs remotely in the cycle of ξ_0. We know that $\epsilon_v = -1$ and $\epsilon_{v+1} = 1$. Hence from the discussion page 148, Perron, Band 1, we see that under the transformation $\mathfrak{T}_1$, the term $\frac{-1|}{|b_v}$ is replaced by $\frac{1|}{|1} + \frac{1|}{|b_v - 1}$ and ξ_v is replaced by $\xi_v/(\xi_v - 1)$ and $\xi_v - 1$, respectively. Then by Lemma 2, section 5.5, as $\xi_v - 1 = \frac{q'+\sqrt{p'^2+q'^2}}{p'}, p' > 2q' > 0$, it follows that the occurrence of a complete quotient such as ξ_v is unique. Similarly in the B.c.f. expansion of $\sqrt{R}/Q_0$, it follows that there is at most one remote complete quotient of the form $\frac{p+q+\sqrt{p^2+q^2}}{p}, p > 2q > 0$.

Examples (AAK).

(a)

$$\xi_0 = \frac{386 + 101 + \sqrt{159197}}{386}, \xi_3 = \frac{374 + 139 + \sqrt{159197}}{374},$$

Here $101^2 + 386^2 = 139^2 + 374^2 = 159197$, period-length=6.

(b)

$$\xi_0 = \frac{82 + 27 + \sqrt{7453}}{82}, \xi_3 = \frac{78 + 37 + \sqrt{7453}}{78},$$

Here $27^2 + 82^2 = 37^2 + 78^2 = 7453$, period-length=6.

We conjecture that if the surds are respectively ξ_0 and ξ_v, where $1 \le v < k$ and k is the period-length, then $v = k/2$. Also $Q_v = Q_{k-v}, 0 \le v \le k$, $P_{k+1-v} = P_v, b_v = b_{k-v}$, $2 \le v \le \frac{k}{2} - 1$ and $b_1 = b_{k-1}, b_{\frac{k}{2}+1} = b_{\frac{k}{2}-1} - 1, b_{\frac{k}{2}} = 2$.