

Generalizations of the $3x + 1$ problem and connections with Markov matrices and chains

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$3x+1$ conjecture (Collatz 1929)

Let $T : \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ (3x + 1)/2 & \text{if } x \text{ is odd.} \end{cases}$$

Then experimentally, the iterates $x, T(x), T^2(x), \dots$

(a) with $x > 0$, reach the cycle 1, 2, 1;

(b) with $x < 0$, reach one of the cycles

$-1, -1$;

$-5, -7, -10, -5$;

$-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34, -17$.

Experiment at

<http://www.numbertheory.org/php/collatz.html>

A generalization

Let a and b be integers, a even, b odd and

$$T(x) = \begin{cases} (x + a)/2 & \text{if } x \equiv 0 \pmod{2}, \\ (3x + b)/2 & \text{if } x \equiv 1 \pmod{2}. \end{cases}$$

We expect all iterates to eventually cycle, with finitely many cycles including the following:

- (i) a, a ;
- (ii) $-b, -b$;
- (iii) $b + 2a, 2b + 3a, b + 2a$;
- (iv) $-5b - 4a, -7b - 6a, -10b - 9a, -5b - 4a$;
- (v) $-17b - 16a, -25b - 24a, -37b - 36a, -55b - 54a,$
 $-82b - 81a, -41b - 40a, -61b - 60a, -91b - 90a,$
 $-136b - 135a, -68b - 67a, -34b - 33a, -17b - 16a.$

The $3x + 371$ mapping

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ (3x + 371)/2 & \text{if } x \text{ is odd.} \end{cases}$$

We believe there are 9 cycles (lengths in parentheses):

$$0 \text{ (1)}, -371 \text{ (1)}, 371 \text{ (2)}, -1855 \text{ (3)}, -6307 \text{ (11)}, \\ 25 \text{ (222)}, 265 \text{ (4)}, 721 \text{ (29)}, -563 \text{ (14)}.$$

Experiment at

<http://www.numbertheory.org/php/3x+371.html>

Hasse's generalization

Let $m > d > 1$, $\gcd(m, d) = 1$ and let R_d be a set of $d - 1$ nonzero residue classes $(\text{mod } d)$. Then

$$T(x) = \begin{cases} x/d & \text{if } x \equiv 0 \pmod{d} \\ (mx - r)/d & \text{if } mx \equiv r \pmod{d}, r \in R_d. \end{cases}$$

Then H. Möller conjectured that the sequence of iterates $x, T(x), T^{(2)}(x), \dots$, eventually cycles for all integers x , if and only if $m < d^{d/(d-1)}$ and that regardless of this inequality, the number of cycles is finite.

Generalized $3x+1$ mappings

Let $d \geq 2$ and m_0, \dots, m_{d-1} be non-zero integers. Also for $i = 0, \dots, d-1$, let $r_i \in \mathbb{Z}$ satisfy $r_i \equiv im_i \pmod{d}$. Then

$$T(x) = \frac{m_i x - r_i}{d} \quad \text{if } x \equiv i \pmod{d}$$

defines a mapping $T : \mathbb{Z} \rightarrow \mathbb{Z}$, called a *generalized $3x+1$ mapping*.

Equivalently, in terms of the integer part symbol,

$$T(x) = \left\lfloor \frac{m_i x}{d} \right\rfloor + a_i \quad \text{if } x \equiv i \pmod{d},$$

where a_0, \dots, a_{d-1} are integers.

kth iterate formula

If $T^k(x) \equiv i \pmod{d}$, $0 \leq i < d$, we define $m_k(x) = m_i$ and $r_k(x) = r_i$. Then

$$(a) \quad T^k(x) = \frac{m_0(x) \cdots m_{k-1}(x)}{d^k} \left(x - \sum_{i=0}^{k-1} \frac{r_i(x) d^i}{m_0(x) \cdots m_i(x)} \right).$$

(b) If $T^i(x) \neq 0$ for all $i \geq 0$, then

$$T^k(x) = \frac{m_0 \cdots m_{k-1}(x)}{d^k} x \prod_{i=0}^{k-1} \left(1 - \frac{r_i(x)}{m_i(x) T^i(x)} \right).$$

Diophantine equation for a cycle

The k th iterate formula (a) gives the following criterion for $x \in \mathbb{Z}$ to start a cycle of length K with odd iterates

$T^{i_t}(x), 0 \leq i_1 < \dots < i_L < K$:

$$(2^K - 3^L)x = \sum_{t=1}^L 2^{i_t} 3^{L-t}. \quad (1)$$

Example. $x = -17$. Here $T^{11}(-17) = -17$ and the iterates

$T^k(-17), 0 \leq k < 11$ are

$-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34$. Hence $i_1 = 0, i_2 = 1, i_3 = 2, i_4 = 3, i_5 = 5, i_6 = 6, i_7 = 7$ and $L = 7$. Then equation (1) gives

$$(2^{11} - 3^7)(-17) = 2363 = 2^0 3^6 + 2^1 3^5 + 2^2 3^4 + 2^3 3^3 + 2^5 3^2 + 2^6 3 + 2^7.$$

Relatively prime maps: Conjectures

Let $\gcd(m_i, d) = 1$ for $0 \leq i \leq d - 1$. (The *relatively prime* case).

- (i) If $|m_0 \cdots m_{d-1}| < d^d$, then all trajectories $\{T^k(x)\}$, $x \in \mathbb{Z}$, eventually cycle.
- (ii) If $|m_0 \cdots m_{d-1}| > d^d$, then almost all trajectories $\{T^k(x)\}$, $x \in \mathbb{Z}$ are divergent (that is, $T^k(x) \rightarrow \pm\infty$).
- (iii) The number of cycles is finite and positive.
- (iv) If the trajectory $\{T^k(x)\}$, $x \in \mathbb{Z}$ diverges, then the iterates are uniformly distributed mod d^α for each $\alpha \geq 1$. i.e.,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{k < N \mid T^k(x) \equiv j \pmod{d^\alpha}\} = \frac{1}{d^\alpha}.$$

An example where $|m_0 \cdots m_{d-1}| < d^d$

$$T(x) = \begin{cases} x/3 & \text{if } x \equiv 0 \pmod{3} \\ (2x - 2)/3 & \text{if } x \equiv 1 \pmod{3} \\ (13x - 2)/3 & \text{if } x \equiv 2 \pmod{3} \end{cases}$$

Here $d = 3$, $m_0 = 1$, $m_1 = 2$, $m_2 = 13$ and $m_0 m_1 m_2 = 26 < 27 = d^d$.

There appear to be six cycles, with starting values $0, 2, 47, -2, -10, -22$.

The trajectory starting with $x = 338$ takes 7161 iterations to reach the cycle beginning with 2. Also the maximum iterate value is $T^{2726}(338)$, a number with 73 digits.

Examples where $|m_0 \cdots m_{d-1}| > d^d$

(1) The $5x + 1$ mapping:

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ (5x + 1)/2 & \text{if } x \text{ is odd,} \end{cases}$$

Here the trajectory starting with $x = 7$ appears to be divergent. There appear to be 5 cycles, with starting values $0, 1, 13, 17, -1$.

(2) (Collatz - a 1-1 map of \mathbb{Z} onto \mathbb{Z}):

$$T(x) = \begin{cases} 2x/3 & \text{if } x \equiv 0 \pmod{3} \\ (4x - 1)/3 & \text{if } x \equiv 1 \pmod{3} \\ (4x + 1)/3 & \text{if } x \equiv 2 \pmod{3} \end{cases}$$

Here the trajectory starting with $x = 8$ appears to be divergent. There appear to be 9 cycles with starting values $0, \pm 1, \pm 2, \pm 4, \pm 44$.

Limiting frequencies conjecture for divergent trajectories (relatively prime T)

For a mapping of relatively prime type, experiments reveal that for each $m > 1$, a divergent trajectory

- (a) eventually belongs to a union $B(j_1, m) \cup \dots \cup B(j_r, m)$, $0 \leq j_1 < \dots < j_r \leq m - 1$ of congruence classes (mod m),
- (b) occupies each $B(j_i, m)$ with a positive limiting frequency f_i ,
- (c) occupies each $B(j_i + tm, md)$, $0 \leq t < d$, with limiting frequency f_i/d .

For a wider class of mappings T , we believe these sets and the frequencies f_i , can be predicted by studying a certain Markov matrix $Q_T(m)$.

An example of limiting frequency behaviour

The $5x - 3$ mapping:

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ (5x - 3)/2 & \text{if } x \text{ is odd,} \end{cases}$$

- (i) $m = 5$. Trajectories such as $\{T^k(-5)\}$ and $\{T^k(-21)\}$ appear to be divergent and eventually occupy the congruence classes $B(1, 5), B(2, 5), B(3, 5), B(4, 5)$ with apparent limiting frequencies $8/15, 1/15, 4/15, 2/15$.
- (ii) $m = 3$. The trajectory $\{T^k(-5)\}$ occupies $B(1, 3)$ and $B(2, 3)$ with apparent limiting frequencies $1/2, 1/2$, whereas the trajectory $\{T^k(-21)\}$ occupies $B(1, 3)$ for all $k \geq 0$.

Size of divergent trajectory k-th iterate

On the assumption that the limiting frequencies for divergent trajectories exist for the classes $B(j, d)$ and equal $1/d$, the product formula for $T^k(x)$ allows us to deduce that

$$|T^k(x)|^{1/k} \rightarrow \frac{|m_0 \cdots m_{d-1}|^{1/d}}{d}.$$

If the limiting frequencies f_i exist, but are not uniform, this limit is replaced by

$$|T^k(x)|^{1/k} \rightarrow \frac{|m_0|^{f_0} \cdots |m_{d-1}|^{f_{d-1}}}{d}.$$

Some properties of T^{-1}

(i) $T^{-1}(B(j, m))$ is a disjoint union of N congruence classes (mod md). Moreover, if $\gcd(m_i, m) = 1$ for $i = 0, \dots, d-1$, then $N = d$.

(ii) In the relatively prime case, the d^α cylinders

$$B(i_0, d) \cap T^{-1}(B(i_1, d)) \cap \dots \cap T^{-(\alpha-1)}(B(i_{\alpha-1}, d)),$$

$0 \leq i_0 < d, \dots, 0 \leq i_{\alpha-1} < d$, are the d^α congruence classes mod d^α .

(iii) In the relatively prime case, if

$$A = B(j, d^\alpha) \text{ and } B = B(k, d^\beta),$$

then $T^{-K}(A) \cap B$ is a disjoint union of $d^{K-\beta}$ congruence classes mod $d^{K+\alpha}$, if $K \geq \beta$.

Extension of T to d -adic integers $\hat{\mathbb{Z}}_d$

We restrict ourselves to the relatively prime case.

T extends uniquely to a continuous mapping $T : \hat{\mathbb{Z}}_d \rightarrow \hat{\mathbb{Z}}_d$. This ring is a compact metric space under the d -adic metric and the "congruence" classes mod d^α form a basis for the open sets.

There is a Haar measure μ on the additive group of $\hat{\mathbb{Z}}_d$, where $\mu(B(j, d^\alpha)) = 1/d^\alpha$.

Property (i) implies that $T^{-1}(B(j, d^\alpha))$ is the disjoint union of d congruence classes (mod $d^{\alpha+1}$); hence T is *measure-preserving*:

$$\mu(T^{-1}(A)) = \mu(A),$$

if A is a measurable set in $\hat{\mathbb{Z}}_d$.

Applying the ergodic theorem to $T : \hat{\mathbb{Z}}_d \rightarrow \hat{\mathbb{Z}}_d$

Property (iii) of T implies the *strongly-mixing* property

$$\lim_{K \rightarrow \infty} \mu(T^{-K}(A) \cap B) = \mu(A)\mu(B)$$

for all measurable sets A and B in $\hat{\mathbb{Z}}_d$; hence T is ergodic:

$$T^{-1}(A) = A \implies \mu(A) = 0 \quad \text{or} \quad 1.$$

Applying the ergodic theorem to $B(j, d^\alpha)$ gives

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{k < N \mid T^k(x) \equiv j \pmod{d^\alpha}\} = \frac{1}{d^\alpha}$$

for almost all $x \in \hat{\mathbb{Z}}_d$.

H. Möller's d -adic expansion for relatively prime T

For all $x \in \hat{\mathbb{Z}}_d$,

$$x = \sum_{i=0}^{\infty} \frac{r_i(x)d^i}{m_0(x) \cdots m_i(x)}.$$

This tells us that the congruence classes mod d occupied by the iterates of x , in fact determine x .

A corresponding expansion is useful in a later example of a mapping $T : GF(2)[X] \rightarrow GF(2)[X]$.

Markov matrix arising from T

To introduce Markov chains, we need a probability space containing \mathbb{Z} , which we take to be the *polyadic integers* $\hat{\mathbb{Z}}$. Like the d -adic integers, this ring is a compact metric space that can be defined as a completion of \mathbb{Z} . The congruence class $\{x \in \hat{\mathbb{Z}} \mid x \equiv j \pmod{m}\}$ is also denoted by $B(j, m)$. Then our finitely additive measure μ on \mathbb{Z} extends to a probability Haar measure on $\hat{\mathbb{Z}}$.

Markov chain equation

Then the sequence of random set-valued functions $Y_K(x) = B(T^K(x), m)$, $x \in \hat{\mathbb{Z}}$, forms a Markov chain with m states $B(j, m)$, $0 \leq j < m$ and transition matrix $Q_T(m) = [q_{ij}(m)]$:

$$\begin{aligned}q_{ij}(m) &= Pr\{(T(x) \in B(j, m) | x \in B(i, m))\} \\ &= \mu\{B(i, m) \cap T^{-1}(B(j, m))\} / \mu\{B(i, m)\}\end{aligned}$$

and Markov property:

$$\begin{aligned}Pr(Y_0(x) = B(i_0, m), \dots, Y_K(x) = B(i_K, m) | Y_0(x) = B(i_0, m)) \\ = q_{i_0 i_1}(m) \cdots q_{i_{K-1} i_K}(m).\end{aligned}$$

Markov chain property continued

This last equation is a translation of the statement:

$B(i_0, m) \cap T^{-1}(B(i_1, m)) \cap \cdots \cap T^{-K}(B(i_K, m))$ consists of $p_{i_0 i_1}(m) \cdots p_{i_{K-1} i_K}(m)$ congruence classes $(\text{mod } md^K)$, where $B(i, m) \cap T^{-1}(B(j, m))$ consists of $p_{ij}(m)$ congruence classes $(\text{mod } md)$.

The equation also holds if $\gcd(m_i, d^2) = \gcd(m_i, d)$ for $0 \leq i < d$, provided d divides m .

If d divides m , a simple formula exists for $q_{ij}(m)$:

$$q_{ij}(m) = \begin{cases} \frac{\gcd(m_i, d)}{d} & \text{if } T(i) \equiv j \pmod{\frac{m}{d} \gcd(m_i, d)}, \\ 0 & \text{otherwise.} \end{cases}$$

A correspondence

With respect to the Markov matrix $Q_T(m)$,

- (a) \mathcal{C} is a *closed* set of states if $B \in \mathcal{C}$ and $q_{BB'} > 0$ imply $B' \in \mathcal{C}$.
- (b) \mathcal{C} is a *positive recurrent* set of states if it is a minimal closed set.

Then under the correspondence

$$S_{\mathcal{C}} = B(j_1, m) \cup \cdots \cup B(j_t, m) \leftrightarrow \mathcal{C} = \{B(j_1, m), \dots, B(j_t, m)\},$$

where $0 \leq j_1 < \cdots < j_t < m$,

- (a) T -invariant sets $S_{\mathcal{C}}$ correspond to closed sets \mathcal{C} ,
- (b) minimal T -invariant sets $S_{\mathcal{C}}$ (*ergodic sets*) correspond to positive recurrent classes \mathcal{C} ,

Structure of the ergodic sets S_C

Let \mathcal{N}_1 be the set of positive integers composed of primes which divide at least one m_i ; also let \mathcal{N}_2 be the set of positive integers which are relatively prime to each m_i .

Also, for $0 \leq i < j < d$ let

$$\Delta_{ij} = r_j(d - m_i) - r_i(d - m_j)$$

and $\Delta = \gcd_{0 \leq i < j < d} \Delta_{ij}$.

Let $S_1^{(m)}, \dots, S_{r(m)}^{(m)}$ be the ergodic sets (mod m). Then the following are all the ergodic sets:

- (a) $\hat{\mathbb{Z}}$ if $m \in \mathcal{N}_2$ and $\gcd(m, \Delta) = 1$;
- (b) $S_1^{(m)}, \dots, S_{r(m)}^{(m)}$, where $m | \Delta$, $m \in \mathcal{N}_2$;
- (c) $S_1^{(m)}$, where $m \in \mathcal{N}_1$;
- (d) any intersection of a set of type (b) and one of type (c).

A mapping property of ergodic sets

Suppose T is a mapping of relatively prime type.

If m divides n and $B(j_1, n) \cup \cdots \cup B(j_t, n)$ is an ergodic set (mod n), then $B(j_1, m) \cup \cdots \cup B(j_t, m)$ is an ergodic set (mod m).

A formula for the stationary distribution $\rho_B, B \in \mathcal{C}$

Let $\rho_{Kij}(m)$ be the number of congruence classes (mod md^K) contained in $B(i, m) \cap T^{-K}(B(j, m))$.

Then the cylinder equation implies

$$[\rho_{Kij}] = [p_{ij}]^K = d^K \{Q_T(m)\}^K.$$

Hence

$$\frac{\mu\{B(i, m) \cap T^{-K}(B(j, m))\}}{\mu\{B(i, m)\}} = \rho_{Kij}(m)/d^K = [\{Q_T(m)\}^K]_{ij}.$$

Then if $B(j, m)$ belongs to \mathcal{C} , by the well-known limit result for Markov matrices, summing over $B(i, m) \in \mathcal{C}$, we get

$$\rho_{B(j, m)} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{K < N} \frac{\mu\{S_{\mathcal{C}} \cap T^{-K}(B(j, m))\}}{\mu\{S_{\mathcal{C}}\}}.$$

Ergodic property

Let \mathcal{C} be a positive recurrent class and for each $B \in \mathcal{C}$, let ρ_B be the component of the unique stationary distribution over \mathcal{C} . Then $S_{\mathcal{C}} = \cup_{B \in \mathcal{C}}$ is T -invariant. Hence an ergodic theorem for Markov chains, applied to the $Y_n(x)$ restricted to $S_{\mathcal{C}}$, gives for a $B \in \mathcal{C}$:

$$\Pr \left(\lim_{K \rightarrow \infty} \frac{1}{K} \#\{n; n < K, Y_n(x) = B\} = \rho_B \right) = 1.$$

In other words,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{k; k < N, T^k(x) \in B\} = \rho_B$$

for almost all $x \in S_{\mathcal{C}}$.

Transient class property

Let \mathcal{P} be the set of positive recurrent states. Then

$$\Pr(Y_n(x) \in \mathcal{P} \text{ for some } n > 0) = 1.$$

Hence we expect all divergent trajectories starting in a *transient* $B(j, m)$ to eventually enter some *ergodic* set S_C , occupying each $B \in S_C$ with limiting frequency $\rho(B)$.

Ergodic sets (mod d)

In the case of relatively prime T , there is only one positive recurrent class, $\mathcal{C}_1 = \{B(0, d), \dots, B(d-1, d)\}$. However for non relatively prime T , where $\gcd(m_i, d^2) = \gcd(m_i, d)$, $0 \leq i < d$, we may have several such classes $\mathcal{C}_1, \dots, \mathcal{C}_r$ (and some transient states). We expect

- (i) if $\prod_{B_j \in \mathcal{C}_i} \left(\frac{|m_j|}{d}\right)^{\rho_{B_j}} < 1$, then all trajectories starting in $S_{\mathcal{C}_i}$ will enter a cycle.
- (ii) if $\prod_{B_j \in \mathcal{C}_i} \left(\frac{|m_j|}{d}\right)^{\rho_{B_j}} > 1$, then almost all trajectories starting in $S_{\mathcal{C}_i}$ will be divergent.

Example 1 of $Q_T(m)$

The $5x - 3$ mapping. Here

$$Q_T(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

There are two positive recurrent classes:

$$\mathcal{C}_1 = \{B(0, 3)\} \text{ and } \mathcal{C}_2 = \{B(2, 3), B(2, 3)\}.$$

The stationary distribution for \mathcal{C}_2 is $1/2, 1/2$. Trajectory $\{T^k(-5)\}$ appears to diverge and occupies $B(1, 3)$ and $B(2, 3)$ with limiting frequencies $1/2, 1/2$. Trajectory $\{T^k(-21)\}$ appears to diverge and occupies $B(0, 3)$ for all $k \geq 0$.

Example 2 of $Q_T(m)$

A four-branched mapping:

$$T(x) = \begin{cases} 3x/2 & \text{if } x \equiv 0 \pmod{4} \\ (x+1)/2 & \text{if } x \equiv 1 \pmod{4} \\ x/2 + 1 & \text{if } x \equiv 2 \pmod{4} \\ (5x+3)/2 & \text{if } x \equiv 3 \pmod{4} \end{cases}$$

$$Q_T(4) = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

on interchanging rows and columns 2 and 3.

Example 2 continued

There are two positive recurrent classes:

$$\mathcal{C}_1 = \{B(0, 4), B(2, 4)\} \text{ and } \mathcal{C}_2 = \{B(1, 4), B(3, 4)\}.$$

The stationary vectors for both classes are $(1/2, 1/2)$. Then

$$\prod_{B_j \in \mathcal{C}_1} \left(\frac{|m_j|}{d} \right)^{\rho_{B_j}} = (3/2)^{1/2} (1/2)^{1/2} < 1,$$

$$\prod_{B_j \in \mathcal{C}_2} \left(\frac{|m_j|}{d} \right)^{\rho_{B_j}} = (1/2)^{1/2} (5/2)^{1/2} > 1.$$

Hence we expect all trajectories starting with an even integer to enter one of the cycles with starting values 0, 2, 4, -8, -32, while most starting with an odd trajectory should diverge or else enter one of the cycles with starting values -1, 1, 3, -5, 7, 79, 87, 103, 107, 123.

Example 3 of $Q_T(m)$

$$T(x) = \begin{cases} x/3 - 1 & \text{if } x \equiv 0 \pmod{3} \\ (x + 5)/3 & \text{if } x \equiv 1 \pmod{3} \\ 10x - 5 & \text{if } x \equiv 2 \pmod{3} \end{cases}$$

$$Q_T(3) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 \end{bmatrix}.$$

$(Q_T(3))^2$ is positive, so all states are positive recurrent, with stationary distribution $1/2, 1/4, 1/4$. Also

$$(1/3)^{1/2}(1/3)^{1/4}(30/3)^{1/4} < 1.$$

Hence we expect all trajectories to eventually cycle. There appear to be five cycles, with starting values $0, 5, 17, -1, -4$.

Example 4 of $Q_T(m)$

$$T(x) = \begin{cases} x & \text{if } x \equiv 0 \pmod{3} \\ (7x + 2)/3 & \text{if } x \equiv 1 \pmod{3} \\ (x - 2)/3 & \text{if } x \equiv 2 \pmod{3} \end{cases}$$

$$Q_T(3) = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

There is one positive recurrent class $\mathcal{C}_1 = \{B(0, 3)\}$ and transient states $B(1, 3)$ and $B(2, 3)$.

Here $3|x$ implies $3|T(x)$; so once a trajectory enters the zero residue class mod 3, it remains there. Experimental evidence (<http://www.numbertheory.org/php/markov.html>) strikingly suggests that if $T^k(x) \equiv \pm 1 \pmod{3}$ for all $k \geq 0$, then the trajectory must eventually enter one of the cycles $-1, -1$ or $-2, -4, -2$. The author offers a \$100 (Australian) prize for a proof. This problem seems just as intractable as the $3x + 1$ problem, but is more spectacular.

The general mapping T

In 1983, George Leigh introduced a Markov chain $\{Y_n\}$, which enabled predictions to be made $(\text{mod } m)$, $d \mid m$, for a wider class of T .

Let $m_i = b_i d_i$, where $b_i \in \mathbb{Z}$, $d_i \in \mathbb{N}$ and $\gcd(d, b_i) = 1$, where d_i divides some power of d , $0 \leq i < d$.

We define a sequence of random functions on $\hat{\mathbb{Z}} : x \rightarrow Y_n(x) \in \mathcal{B}$, the collection of congruence classes of the form $B(j, mk)$, where k divides some power of d :

The random set-valued functions

(a) $Y_0(x) = B(x, m)$;

(b) $Y_{n+1}(x) = B(T^{n+1}(x), mk_{n+1})$, where

$$k_0 = 1, \quad k_{n+1} = \frac{d_j k_n}{\gcd(d_j k_n, d)}$$

and $T^n(x) \equiv j \pmod{d}$, $0 \leq j < d$.

Note that if $\gcd(m_j, d) = 1$ for $0 < j < d$ or $\gcd(m_j, d^2) = \gcd(m_j, d)$ (i.e., $d_j \mid d$) for $0 < j < d$, then $k_n = 1$ for all n .

Markov property and Transition probabilities $q_{BB'}$

We have

$$\begin{aligned} \Pr(Y_0(x) = B_0, \dots, Y_K(x) = B_K | Y_0(x) = B_0) \\ = q_{B_0 B_1} \cdots q_{B_{K-1} B_K}, \end{aligned}$$

where the transition probabilities $q_{BB'}$ are defined as follows:

Let $B = B(j, M)$, $B' = B(j', M')$, $N = Md_j/d$, $N' = \text{lcm}(N, m)$.
Then

$$\begin{aligned} q_{BB'} &= \Pr(Y_{n+1}(x) = B' | Y_n(x) = B) \\ &= \begin{cases} \frac{\gcd(Md_j/m, d)}{d} & \text{if } B' = B(T(j) + tN'/N, N'), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Algorithm for computing the states reached and the $q_{BB'}$

Starting with initial state $B = B(j, M)$ equal to one of $B(0, m), \dots, B(m-1, m)$, form the N'/N states

$$B' = B(T(j) + tN'/N, N'), 0 \leq t < N'/N.$$

These give the states B' with $q_{BB'} > 0$; also $q_{BB'} = N/N'$.

If the process finishes and n states are produced, we get an $n \times n$ transition matrix $Q_T(m)$, for which the row corresponding to state B has N'/N non-zero entries, each equal to N/N' .

Criteria for cycling and divergence

Suppose that the Markov chain for $m = d$ has finitely many states. Also if \mathcal{C} be a positive recurrent class, for each $B \in \mathcal{C}$, let ρ_B be the corresponding limiting probability. Then

- (a) Every divergent trajectory will eventually occupy each class B of some positive class \mathcal{C} , with limiting frequency ρ_B .
- (b) Let \mathcal{C} be a positive recurrent class for the Markov chain (mod d) and let

$$p_j = \sum_{\substack{B \in \mathcal{C} \\ B \subseteq B(j, d)}} \rho_B.$$

Criteria for cycling and divergence continued

Then if

$$\prod_{B(j,d) \in \mathcal{C}} \left(\frac{|m_j|}{d} \right)^{p_j} < 1,$$

all trajectories starting in a $B(j, d) \in \mathcal{C}$ will eventually cycle.

However if

$$\prod_{B(j,d) \in \mathcal{C}} \left(\frac{|m_j|}{d} \right)^{p_j} > 1,$$

almost all trajectories starting in a $B(j, d) \in \mathcal{C}$ will diverge.

Example 1 (Leigh 1983)

Let $T : \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ 12x + 4 & \text{if } x \text{ is odd.} \end{cases}$$

Here $d = 2$, $m_0 = 1$, $m_1 = 24$. Then $d_0 = 1$, $d_1 = 8$ and $\gcd(m_1, d_1^2) = \gcd(24, 4) = 4 \neq \gcd(m_1, d) = 2$.

The recursive scheme for generating the states and positive transition probabilities:

$$\begin{aligned} B(0, 2) &\rightarrow B(0, 2) \\ &\rightarrow B(1, 2) \\ B(1, 2) &\rightarrow B(0, 8) \\ B(0, 8) &\rightarrow B(0, 4) \\ B(0, 4) &\rightarrow B(0, 2). \end{aligned}$$

States: $B(0, 2)$, $B(1, 2)$, $B(0, 8)$, $B(0, 4)$.

Example 1 continued

$$Q_T(2) = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

$Q_T(2)^6 > 0$, so the states $B(0, 2), B(1, 2), B(0, 8), B(0, 4)$ form a positive recurrent class with stationary vector

$$(\rho_{B(0,2)}, \rho_{B(1,2)}, \rho_{B(0,8)}, \rho_{B(0,4)}) = (2/5, 1/5, 1/5, 1/5).$$

Then with $p_0 = \rho_{B(0,2)} + \rho_{B(0,8)} + \rho_{B(0,4)}$ and $p_1 = \rho_{B(1,2)}$,

$$\begin{aligned} (m_0/d)^{p_0} (m_1/d)^{p_1} &= (1/2)^{2/5+1/5+1/5} (24/2)^{1/5} \\ &= (3/4)^{1/5} < 1, \end{aligned}$$

so we expect all trajectories to enter cycles.

An example of Leigh (1986)

$$T(x) = \begin{cases} x/4 & \text{if } x \equiv 0 \pmod{8} \\ (x+1)/2 & \text{if } x \equiv 1 \pmod{8} \\ 20x-40 & \text{if } x \equiv 2 \pmod{8} \\ (x-3)/8 & \text{if } x \equiv 3 \pmod{8} \\ 20x+48 & \text{if } x \equiv 4 \pmod{8} \\ (3x-13)/2 & \text{if } x \equiv 5 \pmod{8} \\ (11x-2)/4 & \text{if } x \equiv 6 \pmod{8} \\ (x+1)/8 & \text{if } x \equiv 7 \pmod{8} \end{cases}$$

We find there are 9 states in the Markov chain mod 8:

$B(0, 8)$, $B(1, 8)$, $B(2, 8)$, $B(3, 8)$, $B(4, 8)$, $B(5, 8)$, $B(6, 8)$, $B(7, 8)$, $B(0, 32)$,

$B(0, 8) \rightarrow B(0; 2; 4; 6, 8)$
 $B(1, 8) \rightarrow B(1; 5, 8)$
 $B(2, 8) \rightarrow B(0, 32)$
 $B(3, 8) \rightarrow B(0; 1; 2; 3; 4; 5; 6; 7, 8)$
 $B(4, 8) \rightarrow B(0, 32)$
 $B(5, 8) \rightarrow B(1; 5, 8)$
 $B(6, 8) \rightarrow B(0; 2; 4; 6, 8)$
 $B(7, 8) \rightarrow B(0; 1; 2; 3; 4; 5; 6; 7, 8)$
 $B(0, 32) \rightarrow B(0, 8)$

An example of Leigh (1986) continued

There are two positive recurrent classes: $\mathcal{C}_1 = \{B(1, 8), B(5, 8)\}$ and $\mathcal{C}_2 = \{B(0, 8), B(0, 32), B(2, 8), B(4, 8), B(6, 8)\}$, with transient states $B(3, 8)$ and $B(7, 8)$.

The limiting probabilities are $\rho_1 = (\frac{1}{2}, \frac{1}{2})$ and $\rho_2 = (\frac{3}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$, respectively.

We have $\rho_1 = \rho_5 = \frac{1}{2}$ and as

$$\prod_{B_j \in \mathcal{C}_1} \left(\frac{|m_j|}{d} \right)^{\rho_j} = (1/2)^{1/2} (3/2)^{1/2} < 1,$$

we expect every trajectory starting in $\mathcal{S}_{\mathcal{C}_1} = B(1, 8) \cup B(5, 8)$ to cycle, reaching one of 1, 13, 61, 205, -11.

An example of Leigh (1986) finished

Also $p_0 = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}$ and $p_2 = p_4 = p_6 = \frac{1}{8}$. Then as

$$\prod_{B_j \in \mathcal{C}_2} \left(\frac{|m_j|}{d} \right)^{p_j} = (1/4)^{5/8} 20^{1/8} 20^{1/8} (11/4)^{1/8} > 1,$$

we expect most trajectories starting in $\mathcal{S}_{\mathcal{C}_2} = B(0, 2)$ to diverge, displaying frequencies $\rho_2 = (\frac{5}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ in the respective component congruence classes. For example, the trajectory starting with 46.

We found 8 cycles lying in $B(0, 2)$, with starting values 0, 10, 158, 3292, 4244, -2, -12, -18.

An example of Venturini (1992)

$$T(x) = \begin{cases} 2500x/6 + 1 & \text{if } x \equiv 0 \pmod{6} \\ (21x - 9)/6 & \text{if } x \equiv 1 \pmod{6} \\ (x + 16)/6 & \text{if } x \equiv 2 \pmod{6} \\ (21x - 51)/6 & \text{if } x \equiv 3 \pmod{6} \\ (21x - 72)/6 & \text{if } x \equiv 4 \pmod{6} \\ (x + 13)/6 & \text{if } x \equiv 5 \pmod{6}. \end{cases}$$

There are 9 states in the Markov chain (mod 6):

$$\begin{aligned} B(0, 6) &\rightarrow B(1, 12), B(5, 12), B(9, 12) \\ B(1, 6) &\rightarrow B(2, 6), B(5, 6) \\ B(2, 6) &\rightarrow B(0, 6), B(1, 6), B(2, 6), B(3, 6), B(4, 6), B(5, 6) \\ B(3, 6) &\rightarrow B(2, 6), B(5, 6) \\ B(4, 6) &\rightarrow B(2, 6), B(5, 6) \\ B(5, 6) &\rightarrow B(0, 6), B(1, 6), B(2, 6), B(3, 6), B(4, 6), B(5, 6) \\ B(1, 12) &\rightarrow B(2, 6) \\ B(5, 12) &\rightarrow B(1, 6), B(3, 6), B(5, 6) \\ B(9, 12) &\rightarrow B(5, 6), \end{aligned}$$

namely $B(0,6), B(1,6), B(2,6), B(3,6), B(4,6), B(5,6), B(1,12), B(5,12), B(9,12)$.

Venturini example finished

The 9 states form a positive recurrent class with limiting probabilities

$$\rho = \left(\frac{18}{202}, \frac{20}{202}, \frac{53}{202}, \frac{20}{202}, \frac{18}{202}, \frac{55}{202}, \frac{6}{202}, \frac{6}{202}, \frac{6}{202} \right).$$

Noting that $B(1,12) \subseteq B(1,6)$, $B(9,12) \subseteq B(3,6)$, $B(5,12) \subseteq B(5,6)$, we get

$$\begin{aligned} p_0 &= \rho_{B(0,6)}, \quad p_1 = \rho_{B(1,12)} + \rho_{B(1,6)}, \quad p_2 = \rho_{B(2,6)}, \\ p_3 &= \rho_{B(9,12)} + \rho_{B(3,6)}, \quad p_4 = \rho_{B(4,6)}, \quad p_5 = \rho_{B(5,12)} + \rho_{B(5,6)}. \end{aligned}$$

Then $\prod_{i=0}^{d-1} (m_i/d)^{p_i} < 1$ and we expect all trajectories to eventually cycle. There appear to be two cycles, with starting values 2 and 6.

<http://www.numbertheory.org/php/venturini1.html>

Example of infinitely many states (Chris Smyth 1993)

$$T(x) = \begin{cases} 3x/2 & \text{if } x \equiv 0 \pmod{2} \\ \lfloor 2x/3 \rfloor & \text{if } x \equiv 1 \pmod{2}. \end{cases}$$

This can be regarded as a 6-branched mapping. The integer trajectories are much simpler to describe than the Markov chain:

- (i) A non-zero even integer $2^r(2c + 1)$ is successively multiplied by $3/2$ until it reaches $3^{r+2}(2c + 1) = 6k + 3$.
- (ii) $6k + 3 \rightarrow 4k + 2 \rightarrow 6k + 3$.
- (iii) $6k + 1 \rightarrow 4k \rightarrow 6k \rightarrow 9k \rightarrow 6k$.
- (iv) $6k + 5 \rightarrow 4k + 3$ and unless we encounter 0 or -1 (fixed points), we must eventually reach $B(1, 6)$ or $B(3, 6)$.

With $m = 6$, there are infinitely many states. e.g.,

$$Y_n(0) = B(0, 2 \cdot 3^{n+1}) \text{ for } n \geq 0.$$

Other rings: $GF(2)[x]$

Here the conjectural picture for trajectories is not so clear. Here is an example of relatively prime type where $|m_0 \cdots m_{|d|-1}| = |d|^{|d|}$, where $|f| = 2^{\deg f}$.

$$T(f) = \begin{cases} \frac{f}{x} & \text{if } f \equiv 0 \pmod{x} \\ \frac{(x^2+1)f+1}{x} & \text{if } f \equiv 1 \pmod{x} \end{cases}$$

Most trajectories appear to cycle. However the trajectory starting from $1 + x + x^3$ exhibits a regularity which enabled its divergence to be proved: If $L_n = 5(2^n - 1)$, then

$$T^{L_n}(1 + x + x^3) = \frac{1 + x^{3 \cdot 2^{n+1}} + x^{3 \cdot 2^{n+2}}}{1 + x + x^2}.$$

The figure next page, shows the first 38 iterates.

Divergent trajectory $\{T^k(1 + x + x^3)\}$ in $GF_2[x]$

The first 38 iterates

```
0:1101
1:11001
2:111101
3:1001001
4:01101101
5:1101101 ←
6:11011001
7:110111101
8:1101001001
9:11001101101
10:111111011001
11:1000010111101
12:01001001001001
13:1001001001001
14:01101101101101
15:1101101101101 ←
16:11011011011001
17:110110110111101
18:1101101101001001
19:11011011001101101
20:1101101111111011001
21:1101101000010111101
22:11011001001001001001
23:110111101101101101101
24:11010011011011011011001
25:11001100110110110111101
26:111111111101101101001001
27:100000001011011001101101
28:01000000100110111111011001
29:100000100110111111011001
30:01000010111101000010111101
31:1000010111101000010111101
32:01001001001001001001001001
33:1001001001001001001001001
34:01101101101101101101101101101
35:1101101101101101101101101 ←
36:11011011011011011011011001
37:110110110110110110110111101
```

Polynomials over GF(2) continued

There are infinitely many cycles, many of which have no recognisable pattern.

However the trajectories starting with

$$g_n = (1 + x^{2^n - 1}) / (1 + x) = 1 + x + \dots + x^{2^n - 2}$$

possess symmetry and are purely periodic, with period-length 2^n .

Cyclic trajectory: $g_4(x) = (1 + x^{15})/(1 + x) \in GF(2)[x]$

```
0:1111111111111111
1:1000000000000011
2:01000000000001111
3:1000000000001111
4:0100000000110011
5:100000000110011
6:0100000011111111
7:100000011111111
8:010000110000011
9:10000110000011
10:010011110001111
11:10011110001111
12:0101100110011001
13:101100110011001
14:001111111111111
15:011111111111111
16:111111111111111
```

Mappings of rings of algebraic integers

Let d be a non-unit in the ring O_K of integers of an algebraic number field K . Then O_K is composed of $|Norm_K(d)|$ congruence classes (mod d) and we can consider generalized $3x + 1$ mappings $T : O_K \rightarrow O_K$. The conjectural picture for trajectories is not entirely clear.

Example 1 (Leigh 1983). $T : \mathbb{Z}[\sqrt{2}] \rightarrow \mathbb{Z}[\sqrt{2}]$ is defined by

$$T(\alpha) = \begin{cases} \alpha/\sqrt{2} & \text{if } \alpha \equiv 0 \pmod{\sqrt{2}} \\ (3\alpha + 1)/\sqrt{2} & \text{if } \alpha \equiv 1 \pmod{\sqrt{2}}. \end{cases}$$

Equivalently, write $\alpha = x + y\sqrt{2}$, where $x, y \in \mathbb{Z}$. Then

$$T(x, y) = \begin{cases} (y, x/2) & \text{if } x \equiv 0 \pmod{2} \\ (3y, (3x + 1)/2) & \text{if } x \equiv 1 \pmod{2}. \end{cases}$$

There appear to be finitely many cycles with starting values

$$0, 1, -1, -5, -17, -2 - 3\sqrt{2}, \quad -3 - 2\sqrt{2}, \quad 9 + 10\sqrt{2}.$$

Example 1 continued

An interesting feature is the presence of at least three one-dimensional T -invariant sets S_1, S_2, S_3 in $\mathbb{Z} \times \mathbb{Z}$:

- (i) $S_1 : x = 0$ or $y = 0$,
- (ii) $S_2 : 2x + y + 1 = 0$ or $x + 4y + 1 = 0$,
- (iii) $S_3 : x + y + 1 = 0$ or $x + 2y + 1 = 0$ or $x + 2y + 2 = 0$.

Trajectories starting in S_1 or S_2 oscillate from one line to the other, while those starting in S_3 oscillate between the first and either of the second and third.

Trajectories starting in S_1 will cycle, as $T^2(x, 0) = (C(x), 0)$ and $T^2(0, y) = (0, C(y))$, where C denotes the $3x + 1$ mapping.

Example 2

$T : \mathbb{Z}[\sqrt{3}] \rightarrow \mathbb{Z}[\sqrt{3}]$ is defined by

$$T(x) = \begin{cases} x/\sqrt{3} & \text{if } x \equiv 0 \pmod{\sqrt{3}} \\ (x-1)/\sqrt{3} & \text{if } x \equiv 1 \pmod{\sqrt{3}} \\ (4x+1)/\sqrt{3} & \text{if } x \equiv 2 \pmod{\sqrt{3}} \end{cases}$$

There are at least 103 cycles. The trajectory starting with $-1 - 5\sqrt{3}$ appears to be divergent. Divergent trajectories produce limiting frequencies approximating $(.27, .32, .40)$ in the residue classes $0, 1, 2 \pmod{\sqrt{3}}$. Interpretation?

Website

- ▶ <http://www.numbertheory.org/php/collatz.html>